Upper Bound Derivation for $C(m,n)$
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In the following it is shown that for $n > 1$ and $m > 3n + 1$, $C(m, n)$ is bounded by

$$C(m,n) < \frac{2m^n}{n!} \tag{1.3.7}$$

From Equation (1.3.4) we may proceed from the expression for $C(m, n)$, for the case $m > n + 1$, as follows

$$C(m,n) = 2\sum_{i=0}^{n}\binom{m-1}{i} = 2\binom{m-1}{n}\left[1+\frac{n}{m-n}+\frac{n(n-1)}{(m-n)(m-n+1)}+\ldots+\frac{n!}{(m-1)(m-2)\ldots(m-n)}\right]$$
$$<2\binom{m-1}{n}\left[1+\frac{n}{m-n}+\frac{n^2}{(m-n)^2}+\ldots\right]$$

Now, assuming $m > 2n$ in the preceding expression, the term in brakets forms a geometric series in $n/(m\text{-}n)$ which allows us to express the preceding inequality as

$$C(m,n) < 2\binom{m-1}{n}\frac{m-n}{m-2n} \tag{1.3.8}$$

Before proceeding further with the derivation, it is noted that in the limit of large n, the preceding expansion converges to the following useful approximation

$$\lim_{\substack{n\to\infty \\ m>2n}} C(m,n) = 2\binom{m-1}{n}\frac{m-n}{m-2n} \tag{1.3.9}$$

This result will be employed in Section 1.5. Getting back to Equation (1.3.8) one may proceed as follows

$$C(m,n) < 2\binom{m-1}{n}\frac{m-n}{m-2n} = 2\frac{(m-1)(m-2)\ldots(m-(n-1))}{n!}\left(\frac{m-n}{m-2n}\right)$$
$$< \frac{2}{n!}m^{n-2}\frac{(m-n+1)(m-n)(m-n)}{m-2n}$$

where $n > 1$ was assumed. Now, since one can show that for $m > 3n + 1$ the rightmost ratio in the preceding expression satisfies the inequality

$$\frac{(m-n+1)(m-n)(m-n)}{m-2n} < m^2$$

one obtains the desired result

$$C(m,n) < \frac{2}{n!}m^{n-2}m^2 = 2\frac{m^n}{n!} \ .$$