Brief notes on Liapunov functions

CNS 185

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1 Motivation

The linear differential equations of basic physics (simple harmonic oscillator, etc.) are usually idealizations of nonlinear systems and as such may not reflect important properties of the original modeled system. In many cases, the behavior of the linear system approximates that of the nonlinear one in some small local region (e.g. a pendulum) and is therefore useful as an approximation. However, many important observed phenomena (limit cycles, chaos) are not possible in linear systems. We know complete solutions to the n-dimensional linear differential equation in constant coefficients, but no such solutions to the general nonlinear problem exist. As a result, researchers beginning with Poincaré have emphasized the *qualitative* analysis of differential equations. With this approach, we make the observation that the long time behavior of a nonlinear differential equation can often be explained without actually solving the equation. Liapunov functions, to be described below, are important tools in this type of analysis.

Informally, we have some kind of *dynamics* described by a (possibly) nonlinear equation[1]

$$
\begin{aligned}
\dot{x_1} &= f_1(x_1, x_2) \\
\dot{x_2} &= f_2(x_1, x_2)
\end{aligned}
\tag{1}
$$

This system of two first-order differential equations might correspond to a two-neuron flip-flop, where the phase space variables x_1 and x_2 are output voltages. A more familiar example might be the dynamics of a 1 degree-of-freedom system moving under forces of gravity, damping, etc. In this case, we must convert from a second-order equation derived from Newton's laws to a system of first order equations (see Homework 1, problem 2). At any rate, the qualitative study of differential equations always begins with conversion to this first-order *phase space* representation, as exemplified by the form of (1).

We will focus on the question of stability at fixed points of (1). Recall that a fixed point in the 2-dimensional phase plane is a point for which $\dot{x_1} = \dot{x_2} = 0$, i.e. if we start the system at this point, the dynamics tells us we remain on this point forever. Stability of a fixed point corresponds nicely with our intuitive notion: a fixed point in phase space is *stable* when starting conditions close to the fixed point remain close for all time. The notion of *asymptotic stability* is a little stronger, and holds for a fixed point when nearby starting conditions not only remain close for all time, but actually move toward the fixed point, approaching it asymptotically. The definition of "close" can be made rigorously, but it corresponds to intuition and is not central here.

Now suppose we had some function $V(x)$ (e.g. $V(x_1, x_2)$ in the planar case) defined over the phase plane near the fixed point we are interested in. Then picking an initial condition and letting the dynamics (1) evolve corresponds to moving around on the surface described by this function. Suppose this function has only one minimum, located at the fixed point $\bar{x}$. If we start at some initial condition near $\bar{x}$, we will observe a trajectory on the surface of our function. If that trajectory leads to decreasing values of V, then we know that the we must approach the minimum of V, i.e. the fixed point $\bar{x}$. But this is just the idea of stability

[1] We write the equation here in 2 dimensions for visualization, but the results hold in n dimensions.

described above, so the very existence of V, with all its stated properties, guarantees the stability of $\bar{x}$. Although general methods for constructing this kind of function do not exist, many systems, particularly those based on physical systems, possess these *Liapunov functions*. Noticing that the nonlinearity of the equations (1) is irrelevant, we can imagine that the study of these functions is a useful tool in nonlinear analysis.

2 Definition

Now we can be a little more concrete about our definition. Remember that in what follows, x represents a vector, the state vector for our system.

Definition: A fixed point solution $\bar{x}$ is **stable** in the sense of Liapunov if for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $||x(t_0) - \bar{x}|| < \delta$ then $||x(t) - \bar{x}|| < \epsilon$ for all $t \geq t_0$. If in addition all neighboring solutions $x(t)$ tend to $\bar{x}$ as t goes to ∞, we say $\bar{x}$ is **asymptotically stable**.

Theorem: Let $\bar{x}$ be an isolated fixed point of the system (1), and let $\bar{x}$ be located in some open set Ω in the phase space of (1). Then V, a differentiable function mapping from the phase space Ω to the real line $\mathcal{R}$, (i.e. a real-valued function) is a **Liapunov function** if it has properties:
1) $V(x) \geq 0$ for all $x \in \Omega$.
2) $V(x) = 0$ if and only if $x = \bar{x}$.
3) The derivative of V with respect to t along solution curves is nonpositive for all t, i.e.:

$$\frac{dV(x(t))}{dt} \leq 0. \tag{2}$$

If a Liapunov function V exists, then $\bar{x}$ is stable. If $\bar{x}$ is the only point in Ω for which $\dot{V} = 0$, then $\bar{x}$ is asymptotically stable. (Note that this last is a sufficient but **not necessary** condition for asymptotic stability.)

You may also notice that this technique can also be used to prove that a fixed point is unstable. Inspection of property 3) above and a little thought will show you that this is the case.

We should emphasize a few points.
• The theorem does not tell you how to construct a Liapunov function; it merely tells you what happens if you *can* construct one.
• Liapunov functions can usually only be found *locally*, that is in the immediate vicinity of some fixed point. In many situations (e.g. a Hopfield net), there will be more than one (locally) stable fixed points, and hence Liapunov functions that exist in many local regions. These regions are often referred to as the *basins of attraction* of the fixed points in question.
• Liapunov functions are never unique, as you can readily imagine multiplying or adding a constant to V without changing the essence of the properties noted above. Beyond this even, there may be sets of functions not related by scalar addition or multiplication that nonetheless all obey the definition above for some particular system.
• There are many other notions of stability, both stronger and weaker, that extend these ideas to more general attractors. We are considering only attractors which are points (have zero dimension). However, attractors of one dimension (limit cycles) or greater (chaos, quasiperiodicity) may exist as well, and are sometimes amenable to the same kind of qualitative analysis.

3 Example

It is illustrative to consider a Liapunov function for a familiar example, the damped pendulum. A back-of-the-envelope diagram and the relation $F = ma$ will convince you that an equation for the dynamics of the damped pendulum is

$$\ddot{\theta} + \gamma\dot{\theta} + \frac{g}{L} sin(\theta) = 0. \tag{3}$$

We can rewrite this as a first order system:

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= -\gamma x_2 - \frac{g}{L} sin(x_1). \end{aligned} \tag{4}$$

The pendulum down position, for which $x_1 = x_2 = 0$, is an isolated fixed point in the region Ω defined by $|x_1| < \pi$.[2]

The equations above describe a dissipative system, in which an initial condition in Ω will evolve, under the dynamics, to the motionless state with the pendulum pointed straight down. This fact follows from our physical intuition that the damping dissipates the potential and kinetic energy that we put into the system (into heat, for example). We can make our intuition more concrete by regarding the total energy of the system as a Liapunov function:

$$V(\mathbf{x}) = \frac{x_2^2}{2} + \frac{g}{L}(1 - cos(x_1)). \tag{5}$$

It would be useful for you to verify, through differentiation, that this does in fact obey the requirements set above. Therefore, we can conclude that the pendulum down position is stable, and is in fact asymptotically stable.

4 References

Almost any book on nonlinear dynamical systems will contain a section on Liapunov functions (also spelled Lyapunov, of Ljapunoff). However, some books are much more accessible than others. Below are three particularly readable books which you will find useful for understanding these concepts. The supplementary reference list for the course also contains useful books, particularly the one by Abraham and Shaw. All the books listed here should be available in the bookstore.

• Beltrami, Ed (1987) *Mathematics for Dynamic Modeling* Academic Press, Boston.
This is a very readable introduction to dynamical systems and the idea of stability in linear and nonlinear systems. Many illustrative examples with physical, visualizable systems (e.g. pendula) are given.

• Hirsch, Morris and Stephen Smale (1974) *Differential equations, dynamical systems, and linear algebra.* Academic Press, Boston.
The Hirsch and Smale text is a classic, and is the ideal starting point for a serious look at dynamical systems. It is written from the geometric and qualitative viewpoint, and is one of the few texts that thoroughly introduces the phase plane analysis of linear differential equations.

• Lasalle, Joseph and Solomon Lefschetz (1961) *Stability by Liapunov's direct method.* Academic Press, New York.
This book deals explicitly with Liapunov methods, and is relatively easy to read.

[2] If we did not restrict ourselves to this "local" region, we would have *two* fixed points to deal with: pendulum up and pendulum down.