

Chapter 8

The Complete Response of RC and RL Circuits

Preview

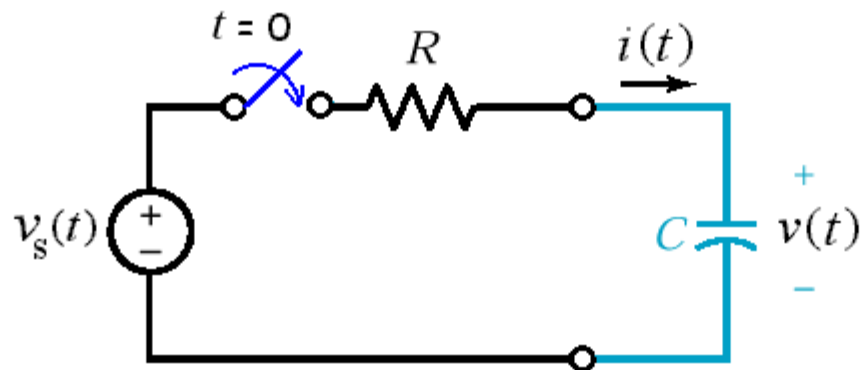
In this chapter, the analysis of switched circuits with resistors, sources, and a single energy storage element (capacitor or inductor) is covered.

Nodal (or mesh) analysis of circuits that contain capacitors and inductors results in differential equations (as opposed to algebraic equations as in the case of resistive circuits).

Linear circuits with a single energy storing element can be represented by a first-order linear differential equation. These circuits are called first order circuits.

The solution for node voltages and branch currents of such circuits requires the solution of the corresponding differential equation.

Response of a first-order RC Circuit



Find $v(t)$ for $t \geq 0$. Assume $v(0)$ is known.

Mesh analysis: $-v_s + Ri + v = 0$ with $i = C \frac{dv}{dt}$

Therefore, $RC \frac{dv}{dt} + v = v_s$

or

$$\boxed{\frac{dv}{dt} + \frac{1}{RC} v = \frac{1}{RC} v_s}$$

First-order linear
differential equation

Equation can be solved if v_s is
specified.

General Solution of a First-Order Differential Equation with Constant Forcing Function (input)

$$\frac{dx}{dt} + ax = f(t) \quad \text{with } f(t) = K \quad (1)$$

(note that: $x = x(t)$)

The complete solution is the sum of the homogeneous solution and the particular solution:

$$x = x_h + x_p$$

Note: in circuits language, x_h corresponds to the natural response, and x_p corresponds to the forced response.

The homogeneous solution is determined as the solution to the differential equation in (1) with $f(t) = 0$,

$$\frac{dx}{dt} + ax = 0 \quad (2)$$

Using the differential operator method, one can find the characteristic equation for (2) which leads to x_h .

General Solution of a First-Order Differential Equation with Constant Forcing Function (input)

The homogeneous solution:

$$\frac{dx}{dt} + ax = 0 \quad (2)$$

Using the differential operator method, one can find the characteristic equation for (2) which leads to x_h .

$\frac{d^2x}{dt^2} \rightarrow s^2$	$\frac{dx}{dt} \rightarrow s$	$x \rightarrow 1$
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Hence, $\frac{dx}{dt} + ax = 0 \rightarrow s + a = 0$ Characteristic Equation

Solving:

$$s = -a$$

$$x_h = Ae^{st} = Ae^{-at}$$

The constant A can be determined from the I.C., as shown later.

General Solution of a First-Order Differential Equation with Constant Forcing Function (input)

The particular solution $x_p = ?$

The particular solution is a function that satisfies the original differential equation in (1). The following table lists the general solution for various $f(t)$

$f(t)$	x_p
K	B
$Ke^{-bt}, b \neq a$	Be^{-bt}
$Ke^{-bt}, b = a$	Bte^{-bt}
$K \sin(\omega t)$ or $K \cos(\omega t)$	$B_1 \cos(\omega t) + B_2 \sin(\omega t)$

For the constant forcing function: $f(t) = K$

The particular solution is a constant: $x_p = B$

Finding B is illustrated in the next slide.

General Solution of a First-Order Differential Equation with Constant Forcing Function (input)

For the constant forcing function: $f(t) = K$
the particular solution is a constant: $x_p = B$

We now substitute $x_p = B$ in the differential equation

$$\frac{dx_p}{dt} + ax_p = K \quad \text{or} \quad 0 + aB = K \quad \text{or} \quad B = \frac{K}{a}$$

Therefore,

$$x_p = \frac{K}{a}$$

General Solution of a First-Order Differential Equation with Constant Forcing Function (input)

Now we are ready to write the complete solution for the first-order differential equation $\frac{dx}{dt} + ax = K$ as

$$x = x_h + x_p = Ae^{-at} + \frac{K}{a}$$

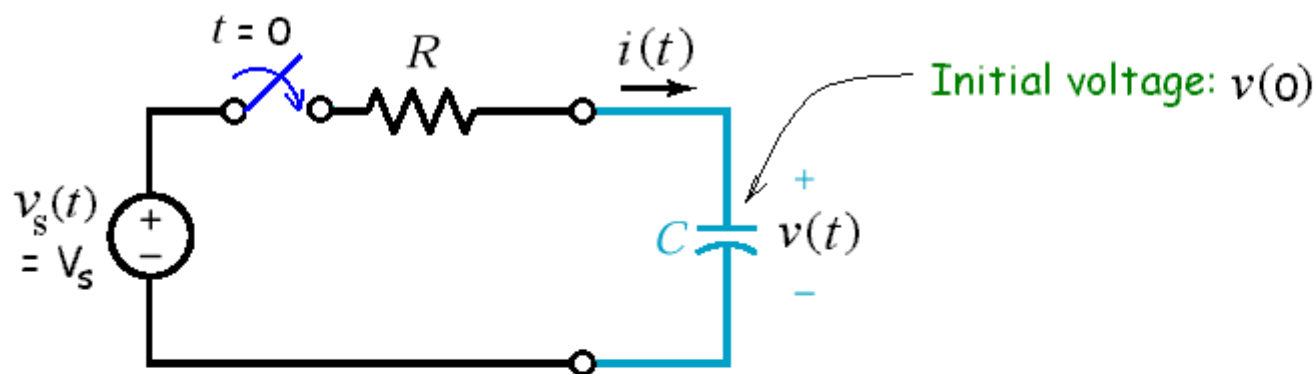
Finally, the initial condition $x(0)$ (assumed known) is used to solve for the constant A ,

$$x(0) = Ae^{-a \cdot 0} + \frac{K}{a} = A + \frac{K}{a}$$

$$\text{or } A = x(0) - \frac{K}{a}$$

The complete solution: $x(t) = \left[x(0) - \frac{K}{a} \right] e^{-at} + \frac{K}{a} \quad \text{for } t \geq 0$

Complete Response for the Series RC Circuit



We have previously determined the following equation for the capacitor voltage:

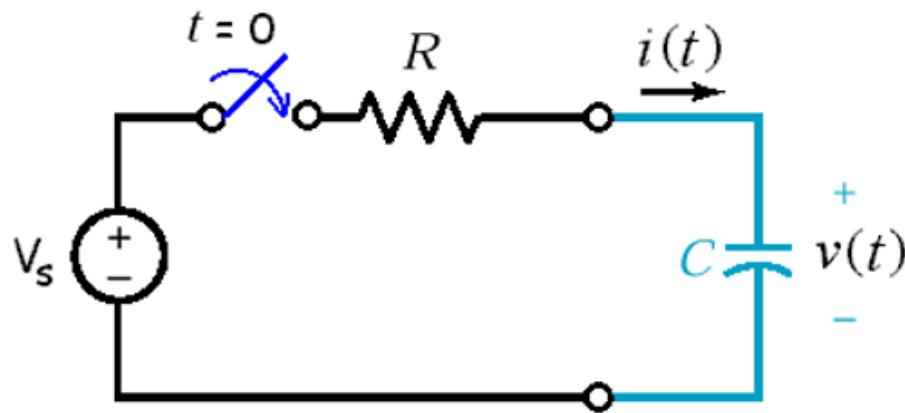
$$\frac{dv}{dt} + \frac{1}{RC}v = \frac{1}{RC}v_s \quad \text{which upon comparing to} \quad \boxed{\frac{dx}{dt} + ax = K}$$

Leads to: $v = x \quad a = \frac{1}{RC} \quad K = \frac{1}{RC}V_s$

$v(t) = [v(0) - V_s]e^{-\frac{t}{RC}} + V_s \quad \text{for } t \geq 0$

$$\boxed{x(t) = [x(0) - \frac{K}{a}]e^{-at} + \frac{K}{a} \quad \text{for } t \geq 0}$$

Complete Response for the Series RC Circuit



$$v(t) = \left[v(0) - V_s \right] e^{-\frac{t}{RC}} + V_s \quad \text{for } t \geq 0$$

$$v(t) = v_n(t) + v_f(t)$$

Natural Response

Decays to zero
for large time.

$$v_n = v_{tr}$$

When the circuit is
stable; i.e., when the
natural frequency is
negative $s < 0$

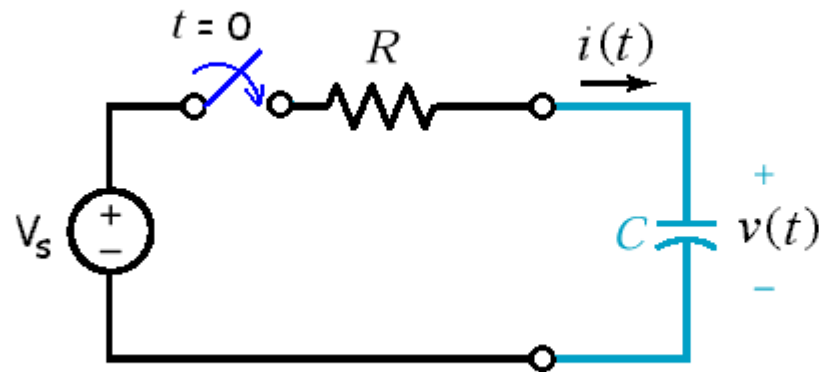
Forced Response

This is the steady-state
response of the circuit.

$$v_f = v_{ss}$$

Natural response: $v_n(t)$
Forced response: $v_f(t)$
Transient response: $v_{tr}(t)$
Steady-state response: $v_{ss}(t)$

Time Constant of an RC circuit

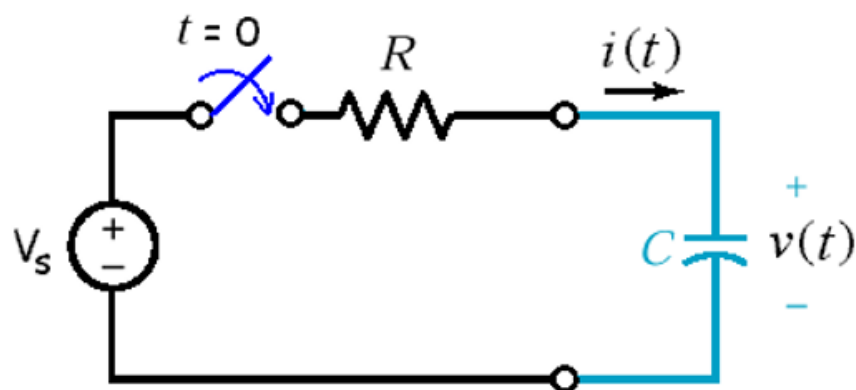


$$v(t) = [v(0) - V_s] e^{\frac{-t}{RC}} + V_s \quad \text{for } t \geq 0$$

The decay rate of the natural response is determined by the constant RC . Note that the unit for RC is:

$$\begin{aligned} (\text{ohm})(\text{Farad}) &= (\text{Volt}/\text{Amp})(\text{Coulomb}/\text{Volt}) = \\ &\text{Coulomb}/\text{Amp} = \\ &\text{Coulomb}/(\text{Coulomb}/\text{sec}) = \boxed{\text{sec}} \end{aligned}$$

Time Constant of an RC circuit



$$v(t) = [v(0) - V_s] e^{-\frac{t}{RC}} + V_s \quad \text{for } t \geq 0$$

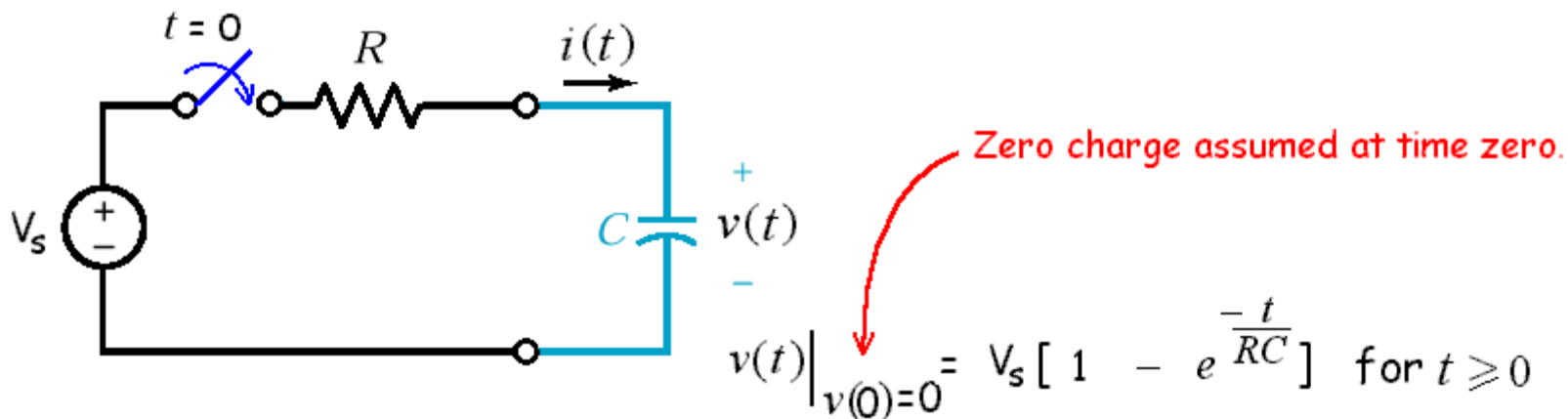
We define the time constant of this circuit as: $\tau = RC$

$$V_s = 0$$

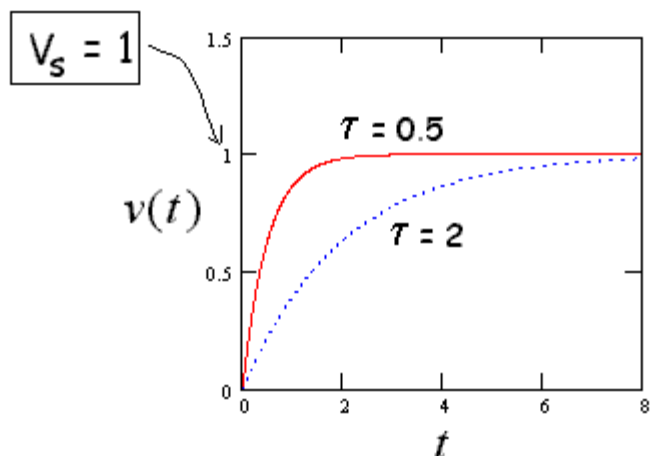
Note That the natural response drops to 0.368 [= $\exp(-1)$] of its initial value for $t = \tau$, and drops by about 95 percent [$\exp(-3) = 0.0498$] for $t = 3\tau$.

The forced response of the series RC circuit [set $v(0) = 0$] is given by $V_s(1 - e^{-\frac{t}{RC}})$. Which indicates that in one time constant the capacitor charges to $1 - \frac{1}{e} = 63.2\%$ of its final value, V_s .

How fast does a Capacitor Charge ?



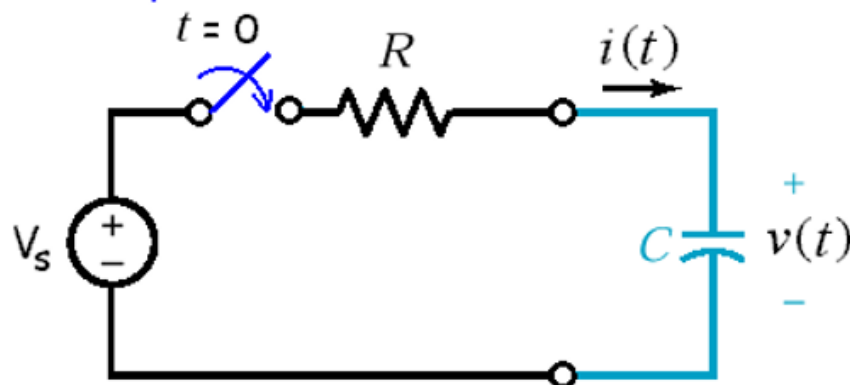
Plots of $v(t)$ for different time constants.



$$\tau = RC$$

Conclusion: The smaller the time constant is (i.e., the smaller R and/or C) the faster the capacitor is to charge.

Complete Solution as Superposition of two responses: one due to the initial voltage across the capacitor and the second is due to the voltage source



$$v(t) = [v(0) - V_s] e^{-\frac{t}{RC}} + V_s \quad \text{for } t \geq 0$$

$$v(t) \Big|_{v(0)=0} = V_s [1 - e^{-\frac{t}{RC}}] \quad \leftarrow \begin{array}{l} \text{Zero-state response } v_{zs}(t) \\ \text{or Zero-initial stored energy} \\ \text{response} \end{array}$$

$$v(t) \Big|_{V_s=0} = v(0) e^{-\frac{t}{RC}} \quad \leftarrow \begin{array}{l} \text{Zero-input response } v_{zi}(t) \end{array}$$

Note that the principle of superposition is satisfied, since

$$v(t) = v(t) \Big|_{v(0)=0} + v(t) \Big|_{V_s=0} = v_{zs}(t) + v_{zi}(t)$$

Example: RC Circuit Complete Response

Find the capacitor voltage after the switch opens in the following circuit. Assume that the switch was closed for a very long time.

Solution:

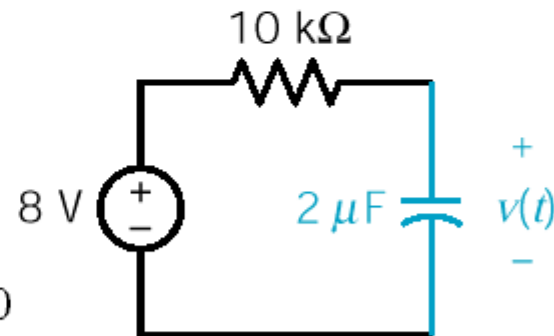
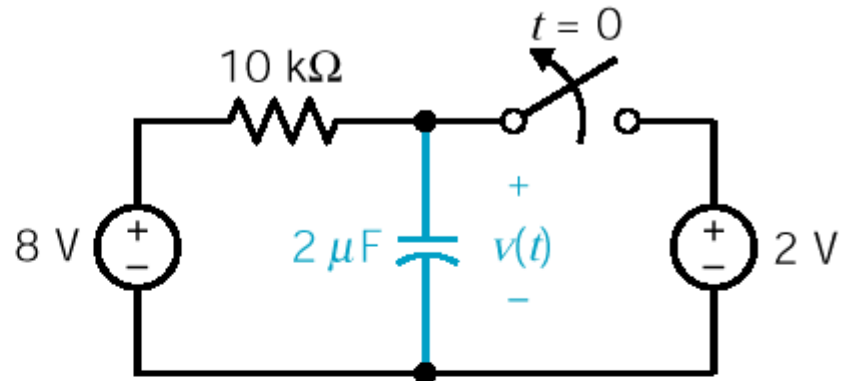
At $t = 0^-$, the capacitor is connected in parallel with the 2 V source. Therefore, $v(0^-) = 2$ V.

The circuit for $t > 0$ is a standard series RC circuit. Therefore, the complete solution is given by

$$v(t) = [v(0) - V_s] e^{-\frac{t}{RC}} + V_s \quad \text{for } t \geq 0$$

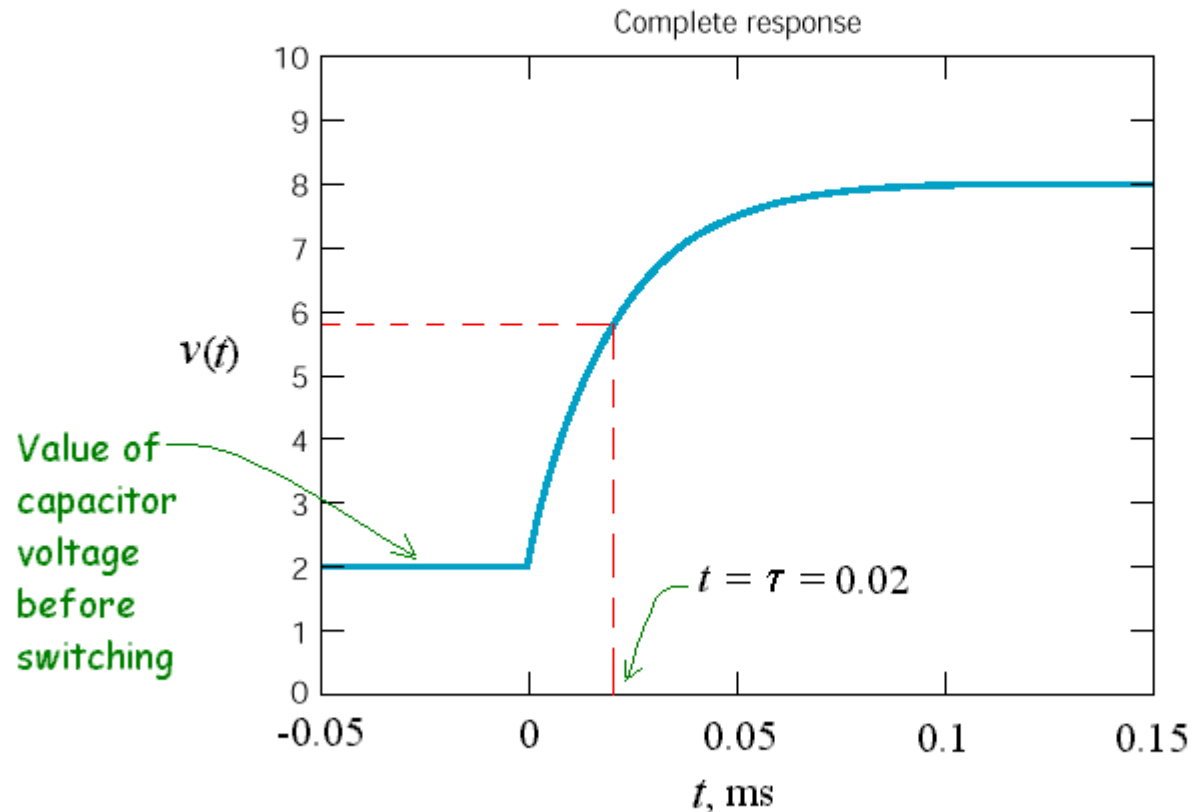
Now because of capacitor voltage continuity property, we have $v(0^+) = v(0) = v(0^-) = 2$ V. Also, $V_s = 8$ and $RC = 0.02$ s.

Therefore, $v(t) = 8 - 6e^{-\frac{t}{0.02}}$ Volts for $t \geq 0$

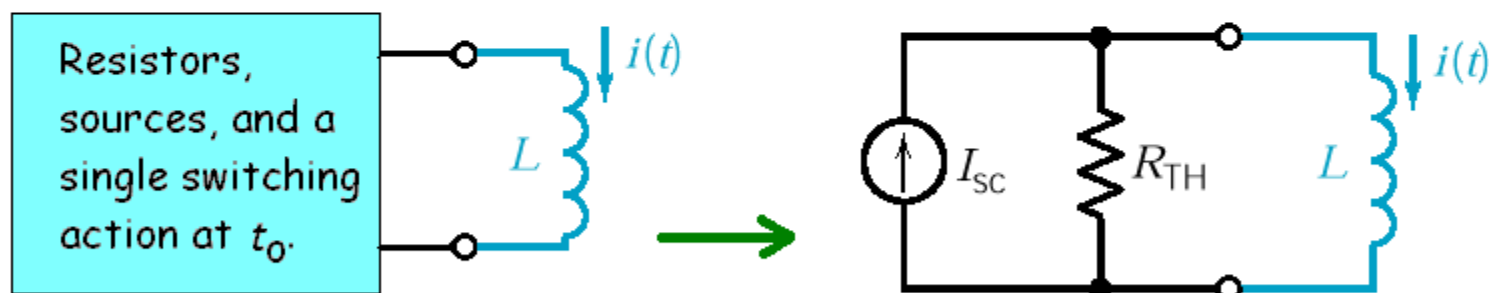
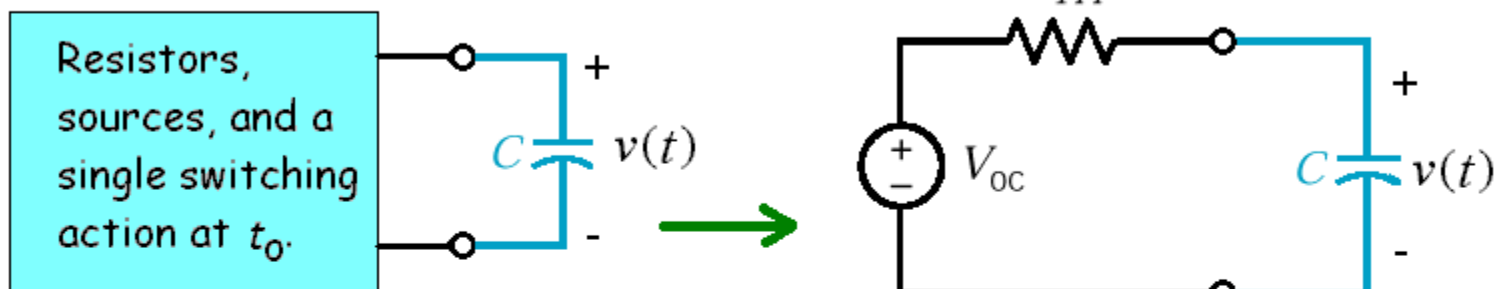


Example: RC Circuit Complete Response

$$v(t) = 8 - 6e^{-\frac{t}{0.02}} \text{ Volts for } t \geq 0$$



Thevenin (or Norton) Analysis Simplifies first-Order RC Circuit Analysis



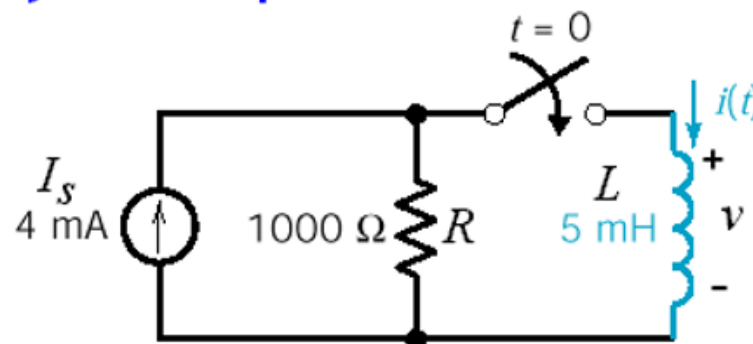
$$\frac{dx}{dt} + ax = f(t)$$

Both equivalent circuits are described by the first-order differential equation on the left, with $x = v(t)$ or $i(t)$.

First-Order Circuits Analysis Examples

Example #1:

Find the inductor current after the switch closes in the following circuit. How long does it take for the inductor current to reach 2 mA?



Solution: First, note that $i(0^+) = i(0^-) = 0$ A.

Apply KCL at top node: $\frac{v}{R} + i = I_s$ with $v = L \frac{di}{dt}$

which leads to $\frac{L}{R} \frac{di}{dt} + i = I_s$ or $\frac{di}{dt} + \frac{R}{L} i = \frac{R}{L} I_s$

Compare to the following equation and its associated solution

$$i(t) = [i(0) - I_s] e^{-\frac{R}{L}t} + I_s \text{ for } t \geq 0$$

$$\frac{dx}{dt} + ax = K$$

(we define $\tau = \frac{L}{R}$)

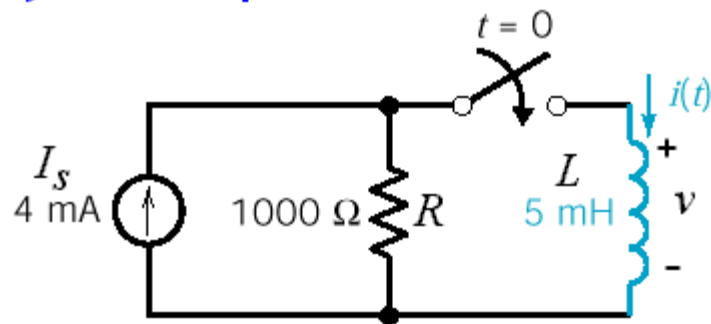
The unit for L/R is
 $\frac{\text{H}}{\Omega} = \frac{\frac{(\text{Volt})(\text{sec})}{\text{Amp}}}{\frac{\text{Volt}}{\text{Amp}}} = \text{sec.}$

$$x(t) = [x(0) - \frac{K}{a}] e^{-at} + \frac{K}{a} \text{ for } t \geq 0$$

First-Order Circuits Analysis Examples

Example #1: (cont.)

Find the inductor current after the switch closes in the following circuit. How long does it take for the inductor current to reach 2 mA?



Solution:

$$i(0^+) = i(0^-) = i(0^-) = 0\text{ A.}$$

$$\tau = \frac{L}{R} = 0.005/1000 = 5 \times 10^{-6} = 5\ \mu\text{s}$$

$$i(t) = 4 - 4e^{-\frac{t}{5}}\text{ mA for } t \geq 0 \text{ (time } t \text{ is in } \mu\text{s.)}$$

To find the time when the current reaches 2 mA, substitute $i(t)=2\text{ mA}$. Then

$$2 = 4 - 4e^{-\frac{t}{5}}\text{ mA}$$

$$\text{Solving for } t \text{ gives } t = -5 \ln \frac{2-4}{-4} = 3.47\ \mu\text{s.}$$

$$i(t) = [i(0) - I_s]e^{-\frac{R}{L}t} + I_s \text{ for } t \geq 0$$

On Energy and the Time Constants RC and L/R

The unit for the quantities RC and L/R is that of time (seconds) The exponential term $e^{-\frac{t}{\tau}}$ that appears in the response of RC (with $\tau = RC$) and RL (with $\tau = L/R$) circuits is independent of the input source. It is referred to as the *natural mode* of the circuit. If we assume no source driving the circuit and that the circuit starts with some initial voltage across the capacitor, $v_C(0)$, (or the inductor's initial current is $i_L(0)$) then the response of the RC and RL circuits will be given by, respectively,

$$v_C(t) = v_R(t) = v_C(0)e^{-\frac{t}{RC}} \quad \text{and} \quad i_L(t) = i_R(t) = i_L(0)e^{-\frac{t}{L/R}}$$

The initial energies are $w_C(0) = \frac{1}{2}Cv_C^2(0)$ for the capacitor and $w_L(0) = \frac{1}{2}Li_L^2(0)$ for the inductor. The energy dissipated in the resistor at time τ (i.e., after the circuit operates for one time constant, τ) will be

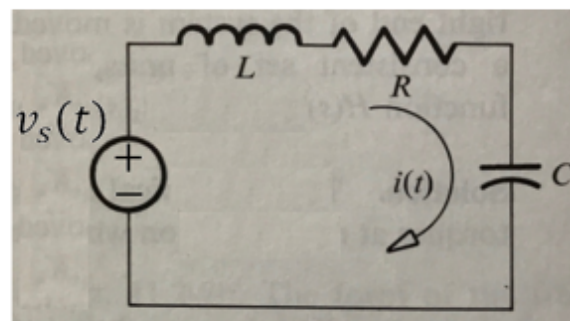
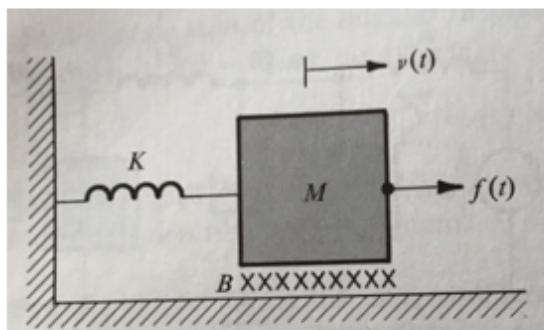
$$RC \text{ circuit: } w_C(\tau) = \frac{1}{2}Cv_C^2(0) - \frac{1}{2}C[v_C(0)e^{-1}]^2 = \frac{1}{2}Cv_C^2(0) \left(1 - \frac{1}{e^2}\right) \cong 87\% \text{ of } w_C(0)$$

$$RL \text{ circuit: } w_L(\tau) = \frac{1}{2}Li_L^2(0) - \frac{1}{2}L[i_L(0)e^{-1}]^2 = \frac{1}{2}Li_L^2(0) \left(1 - \frac{1}{e^2}\right) \cong 87\% \text{ of } w_L(0)$$

Analogy between RC, RL and Mechanical Circuits

Consider the translational mechanical system (circuit) shown below where an object connected to a linear spring exhibits viscous friction as it is moved due to a force $f(t)$ acting on the object. Let M be the mass of the object, $v(t)$ be its velocity, K be the spring constant and B is viscous friction (due to a thin film of oil placed between the object and the track). From Newton's second law of motion, we have the following governing equation:

$$M \frac{dv}{dt} + Bv(t) + K \int v(t)dt = f(t)$$



An electric circuit that is analogous to the above mechanical system is shown above. The circuit is governed by the integral-differential equation (by KVL):

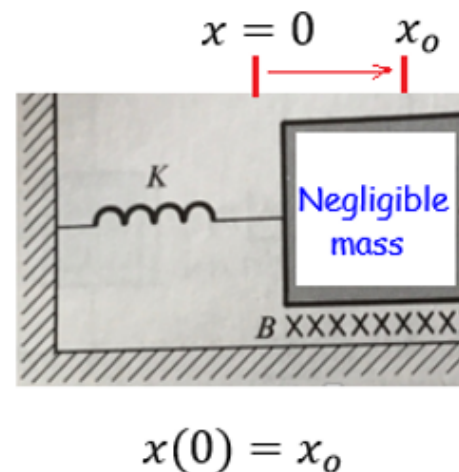
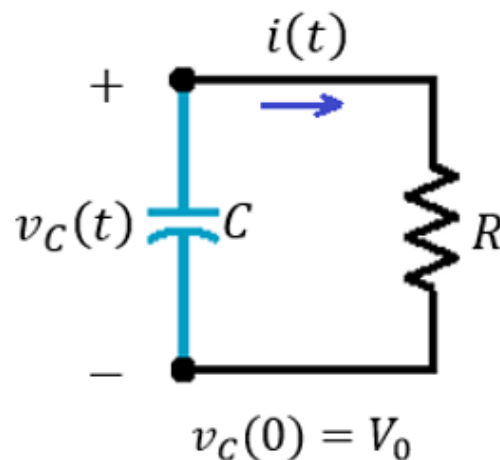
$$L \frac{di(t)}{dt} + Ri(t) + \frac{1}{C} \int i(t)dt = v_s(t)$$

Analogy between RC, RL and Mechanical Circuits (cont.)

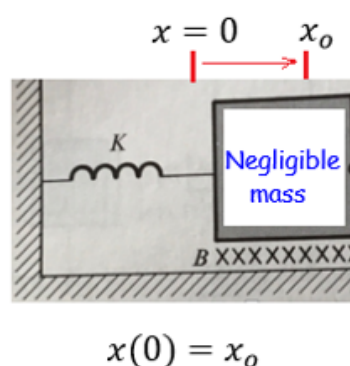
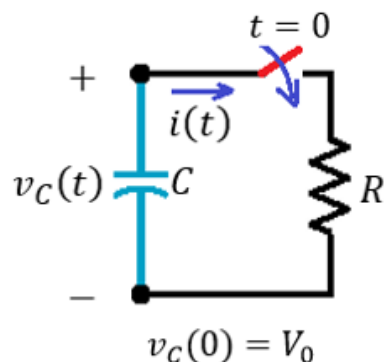
By comparing the above equations we observe that the following quantities are the analog of each other

$$M \leftrightarrow L \quad B \leftrightarrow R \quad K \leftrightarrow \frac{1}{C} \quad v(t) \leftrightarrow i(t) \quad f(t) \leftrightarrow v_s(t)$$

Therefore, the mechanical analog of the source-free RC circuit (with initial capacitor voltage) is as depicted below. The mechanical system has no external force $f(t)$ acting on it. We assume it has an initial potential energy (stored in the spring) due to an initial displacement of the mass to the right of the equilibrium point. Therefore, both circuits store initial potential energy of $\frac{1}{2}CV_0^2$ (electrical potential) and $\frac{1}{2}Kx_0^2$ (elastic energy).



Analogy between RC, RL and Mechanical Circuits (cont.)



The initial capacitor's electric potential energy ($\frac{1}{2}CV_0^2$) is dissipated in the resistor after the switch is closed. This is analogous to the situation where, in the mechanical system, the object is let go and the elastic potential energy in the spring is consumed by friction. We have assumed that the object has negligible mass so that no energy is converted to kinetic energy in the mass. This is analogous to the assumption that the circuit has no inductance.

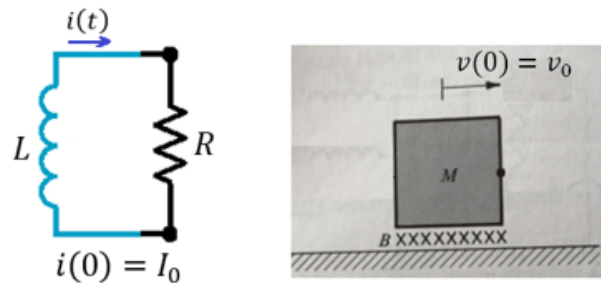
We would expect that as the viscous resistance B increases in value, the slower will be the rate of dissipation of the potential energy; the mass will be moving slower thus preserving potential energy for longer time.

Equivalently, for the RC circuit, we would expect that the energy stored in the capacitor will dissipate in the resistor at a slower rate as the resistance R is increased. We can see that mathematically from the

expression of *capacitor stored energy at time t* , $w_C(t) = \frac{1}{2}CV_0^2 \left(e^{-\frac{2}{RC}t} \right)$: The larger R is, the larger is the term $e^{-\frac{2}{RC}t}$ is and, in turn, the larger the remaining capacitor energy is going to be.

Analogy between RC , RL and Mechanical Circuits (cont.)

A source-free RL circuit [with initial current $i_L(0) = I_0$] and its equivalent mechanical system are shown below. The system has no spring attached and the mass is assumed to have an initial velocity $v(0) = v_0$.



The initial inductor's energy ($\frac{1}{2}LI_0^2$) is dissipated in the resistor. This is analogous to the situation where, in the mechanical system, the object's kinetic energy ($\frac{1}{2}Mv_0^2$) is consumed by viscous substance. We have assumed that the object is not connected to a spring. This is analogous to the assumption that the circuit has no capacitance.

We would expect that as the viscous resistance B increases in value, the faster will be the rate of dissipation of the potential energy. It is like you apply the brakes harder your car will stop quicker. Equivalently, for the RL circuit, we would expect that the (magnetic flux) energy stored in the inductor (analogous to kinetic energy) will dissipate in the resistor more quickly as the resistance R is increased. We can see that mathematically in the expression of *stored inductor energy at time t* , $\frac{1}{2}LI_0^2 \left(e^{-\frac{2R}{L}t} \right)$: The larger R is, the smaller is the term $e^{-\frac{2R}{L}t}$ is and, in turn the smaller the remaining inductor energy is going to be. (Can you see why the time constant for an RL circuit is L/R and not RL ?)

First-Order Circuits Analysis Examples

Example #2:

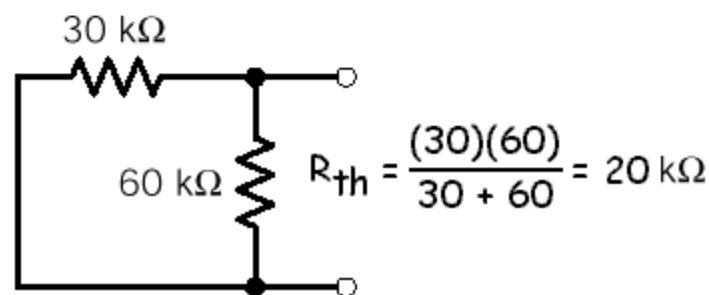
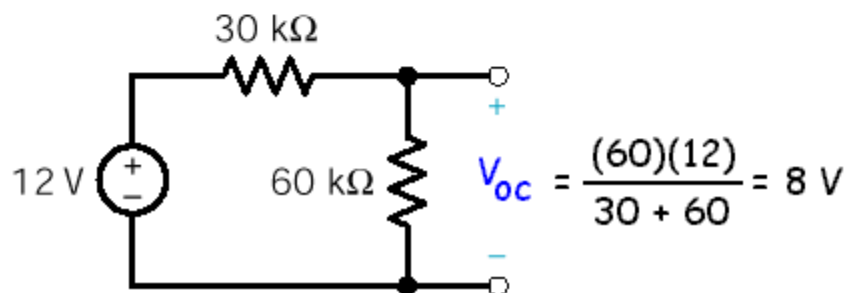
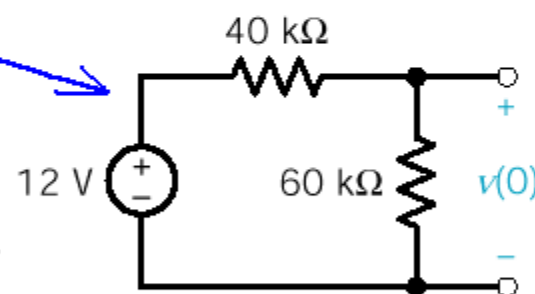
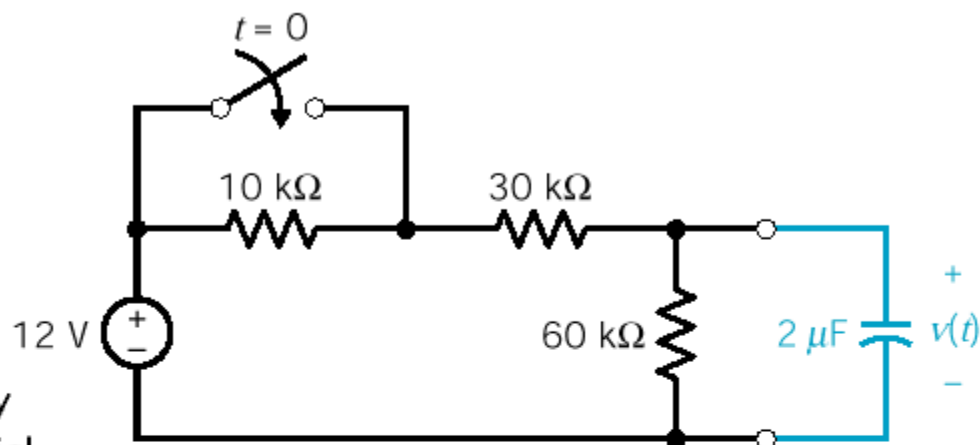
Find the capacitor voltage after the switch closes. Assume that the circuit is in steady-state before the switch closes.

Solution:

Before the switch closes and by voltage division, we find the initial capacitor voltage as

$$v(0^-) = (60)(12)/(10 + 30 + 60) = 7.2 \text{ V}$$

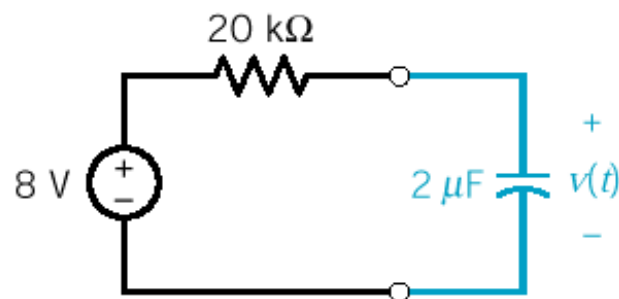
For positive time, the switch is closed and the 10k resistor is shorted. V_{oc} and R_{th} are determined from the following two circuits, respectively.



First-Order Circuits Analysis Examples

Example #2: (cont.)

Therefore, the circuit may be replaced by its Thevenin equivalent circuit as shown here.
with $v(0^-) = 7.2$ V.



We already know the solution for the capacitor voltage for such a series RC circuit with a constant voltage source

$$v(t) = [v(0) - V_s] e^{-\frac{t}{RC}} + V_s \quad \text{for } t \geq 0$$

with $V_s = V_{oc}$ and $R = R_{th}$.

Hence, the solution after substituting all known values is given by

$$v(t) = 8 - 0.8e^{-\frac{t}{.04}} \text{ volts} \quad \text{for } t \geq 0 \text{ (time measured in sec)}$$

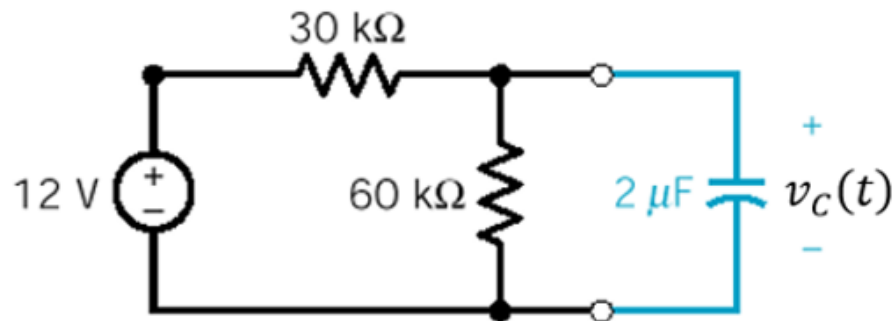
Note that circuits with energy storing elements and resistors are always stable. This is because the time constant $\tau = RC$ is always positive and therefore leads to a decaying natural response due to the $\exp(-t/RC)$ term.

The Direct Analysis Method

Example #2 revisited

The direct analysis method consists of the direct application of nodal and/or mesh analysis for finding the differential equation describing a circuit element voltage or current. It does not use formulas to solve the differential equation; it involves solving the differential equation starting from basic principles. The initial conditions are obtained from the circuit at $t = 0^-$ and 0^+ .

As we did earlier, the initial condition was determined: $v_C(0^+) = 7.2\text{V}$. Next, we analyze the circuit at $t > 0$ using nodal analysis:



$$\frac{v_C - 12}{30,000} + \frac{v_C}{60,000} + (2)10^{-6} \frac{dv_C}{dt} = 0 \Rightarrow \frac{dv_C}{dt} + 25v_C = 200$$

The Direct Analysis Method

Example #2 revisited

The next step is to solve the differential equation:

$$\frac{dv_C}{dt} + 25v_C = 200 \text{ with } v_C(0) = 7.2$$

The forcing function is constant (200). Therefore, the forced response is $v_f(t) = B$. Substituting $v_f(t)$ in the differential equation allows us to solve for B ,

$$0 + 25B = 200 \rightarrow B = 8$$

The characteristic equation is $s + 25 = 0$. Therefore, the natural frequency of the circuit is $s = -25$, leading to the natural response,

$$v_n(t) = Ae^{-25t}$$

Applying the initial condition $v_C(0) = 7.2$ to the complete response:

$$v_C(0) = v_n(0) + v_f(0) = Ae^{-25(0)} + 8 = A + 8 = 7.2 \rightarrow A = -0.8$$

Finally,

$$v_C(t) = 8 - 0.8e^{-25t}, \quad t > 0$$

Your turn: Repeat the analysis using the direct method

First-Order Circuits Analysis Examples

Example #3:

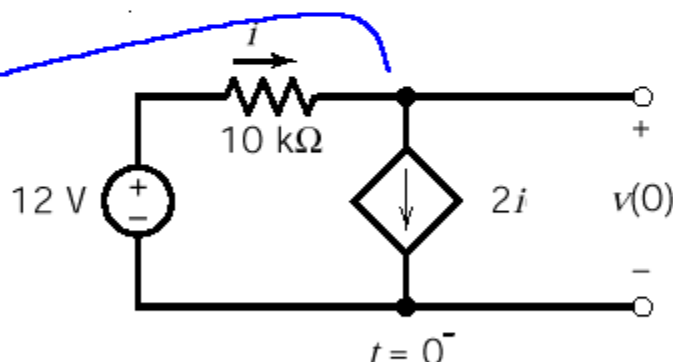
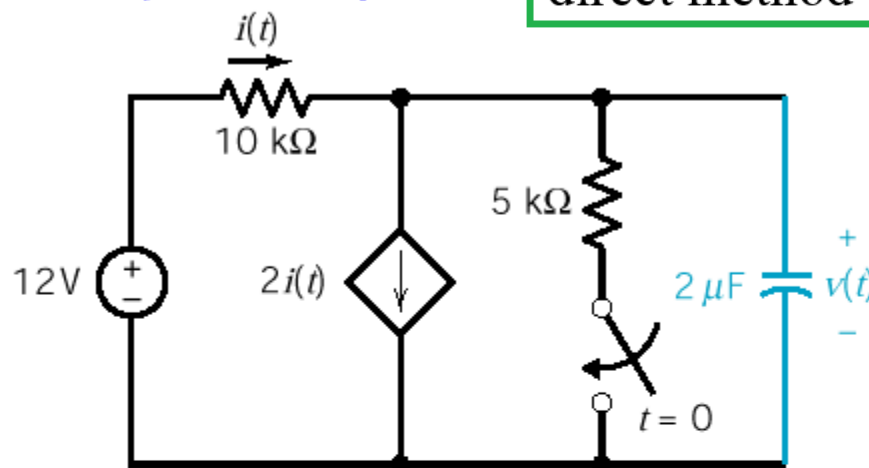
Find the capacitor voltage for positive time. Assume circuit is in steady-state before closing the switch.

Solution:

Since the input to the circuit is a constant, then the capacitor acts as open circuit in steady-state.

KCL at top node: $-i + 2i = 0$ or $i = 0$.

Therefore, there is no voltage drop across the resistor. Hence, $v(0^-) = 12\text{ V}$



Next, we determine the Thevenin equivalent circuit for the part of the circuit connected to the capacitor. This requires two computations. One for the open-circuit voltage and the other for the Thevenin resistance. These computations are illustrated next.

First-Order Circuits Analysis Examples

Example #3: (cont)

KVL for the loop consisting of the two resistors and the voltage source

$$10000i + 5000(i - 2i) = 12$$

or $i = 2.4 \text{ mA}$.

Then, using Ohm's law:

$$V_{oc} = 5000(i - 2i) = -5000i = -12 \text{ V}$$

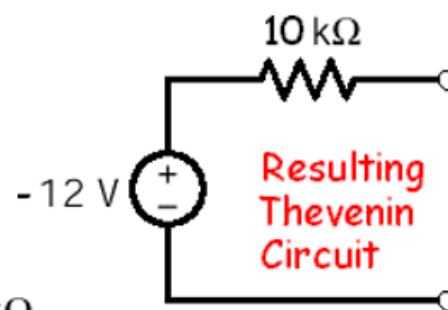
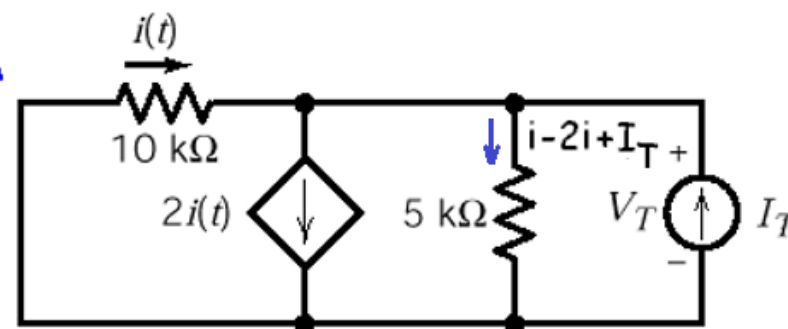
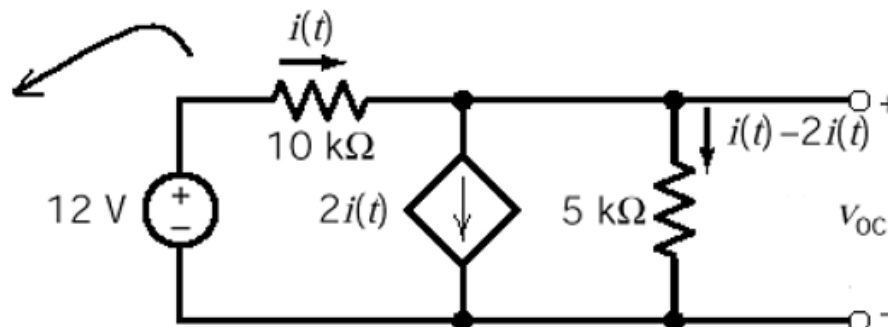
R_{th} is determined by inserting a constant current source I_T at the output node of the circuit and by deactivating the independent source

Now, apply KVL to the loop consisting of the two resistors

$$10000i + 5000(i - 2i + I_T) = 0 \text{ or } i = -I_T$$

Applying Ohm's law to the 5K resistor, we get $V_T = 5000(I_T + i - 2i) = 10000 I_T$

The Thevenin resistance is $R_{th} = V_T / I_T = 10 \text{ k}\Omega$



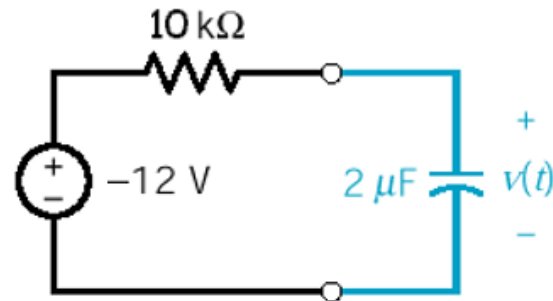
Resulting
Thevenin
Circuit

Try nodal analysis
to find R_{th}

First-Order Circuits Analysis Examples

Example # 3: (cont)

From the preceding analysis we arrive at the reduced series RC circuit shown to the right.



Recalling that the initial capacitor voltage was 12 V, and the time constant $RC = 0.02 \text{ s}$ we can substitute these values in the expression for the complete response that was derived earlier to get

$$v(t) = [v(0) - V_s] e^{-\frac{t}{RC}} + V_s \quad \text{for } t \geq 0$$

$$\text{or } v(t) = [12 - (-12)] e^{-\frac{t}{0.02}} + (-12) \quad \text{for } t \geq 0$$

$$\text{or } v(t) = -12 + 24e^{-50t} \quad \text{for } t \geq 0$$

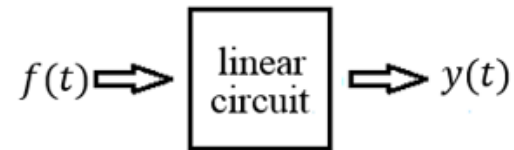
Verification: $v(0) = 12$ and $v(\infty) = -12$

The presence of the dependent source in an RC circuit can lead to an unstable circuit, where $\lim_{t \rightarrow \infty} v(t) = \infty$. Study Examples 8.5-1 and 8.5-2 (pp. 340-342) in your textbook (Svoboda, 9th edition).

Linearity Revisited:

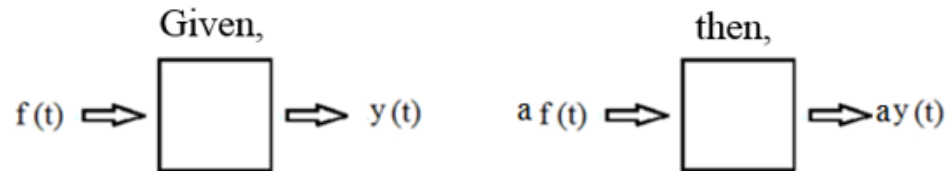
Additional Properties of Linear Circuits

Consider the following linear circuit that responds with $y(t)$ (a current or voltage signal) to the input signal $f(t)$:

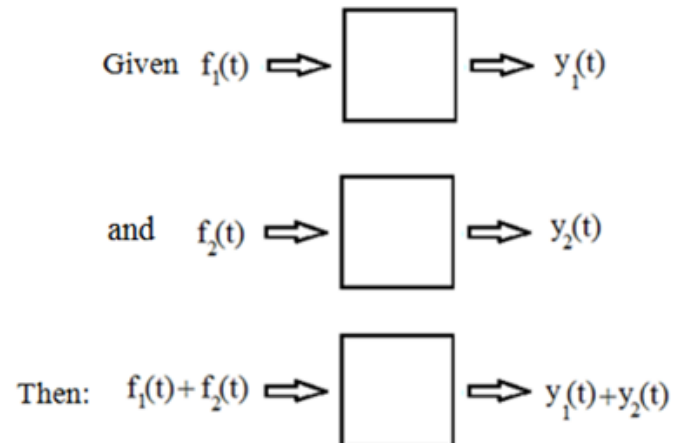


Properties of Linear Circuits:

(i) Homogeneity (scaling) Property



(ii) Superposition:



Linearity Revisited:

Additional Properties of Linear Circuits

Additional Properties of Linear Systems

Assume $y(t)$ is given as the response of the linear system for input $f(t)$.



Time-Shift:

$$f(t - a) \rightarrow \boxed{} \rightarrow y(t - a)$$

$a > 0$

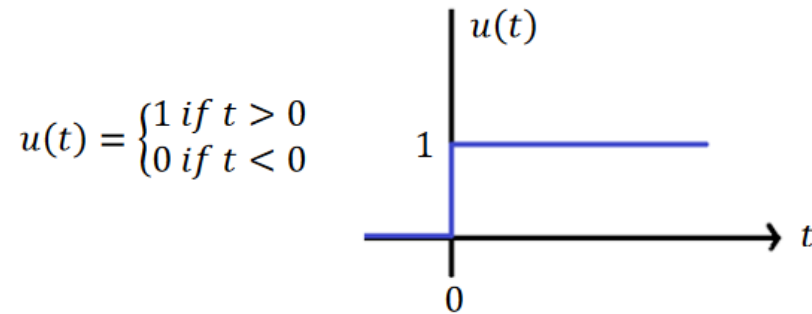
Differentiation:

$$\frac{df(t)}{dt} \rightarrow \boxed{} \rightarrow \frac{dy(t)}{dt}$$

Integration:

$$\int_{0^-}^t f(\tau) d\tau \rightarrow \boxed{} \rightarrow \int_{0^-}^t y(\tau) d\tau$$

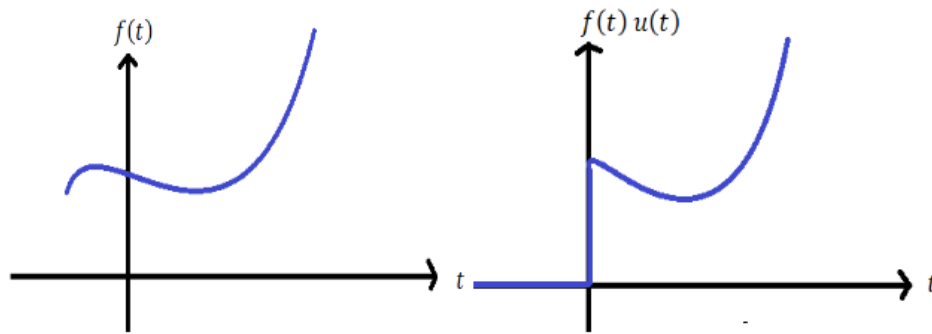
The Unit-Step Function



Mathcad employs the symbol $\Phi(t)$ for the unit-step function, with $\Phi(0) = 1$. So, Mathcad assumes $u(0) = 1$.

Multiplication of $f(t)$ by $u(t)$ leads to a signal:

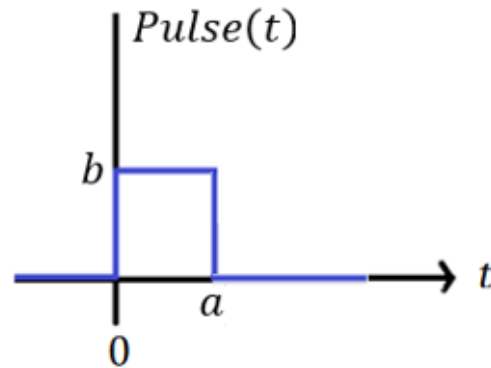
$$f(t)u(t) = \begin{cases} f(t) & t > 0 \\ 0 & t < 0 \end{cases}$$



The Heaviside step function, or the unit step function, usually denoted by $u(t)$, was introduced by Oliver Heaviside (a British Physicist, mathematician and pioneer electrical engineer; 1850-1925). The function was originally developed in **operational calculus** for the solution of differential equations. Oliver Heaviside, who developed the operational calculus as a tool in the analysis of telegraphic communications, represented the function as 1.

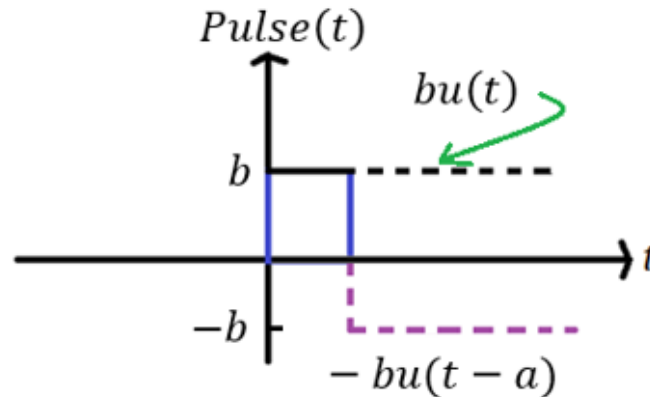


The Pulse Function



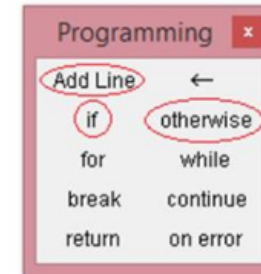
Representation of $Pulse(t)$ using unit-step functions:

$$Pulse(t) = b[u(t) - u(t - a)] = bu(t) - bu(t - a)$$



Mathcad Representation of Piece-Wise Signals

$$f(t) := \begin{cases} -4e^{-t} & \text{if } 0 \leq t < 1 \\ 2t - 4 & \text{if } 2 \leq t < 4 \\ 0 & \text{otherwise} \end{cases}$$

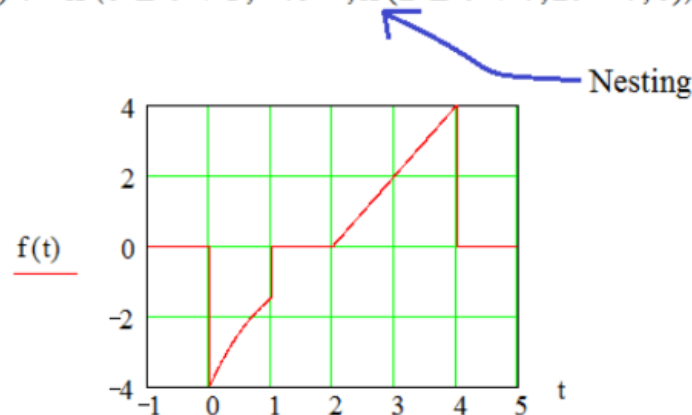


Use the right bracket key “]” or “Add Line” from the Programming Menu to insert the vertical line. Also, “if” and “otherwise” are obtained from the Programming Menu.

Alternatively, one may express $f(t)$ employing the “if” statement:

if(domain, function in domain, function outside domain)

$$f(t) := \text{if}(0 \leq t < 1, -4e^{-t}, \text{if}(2 \leq t < 4, 2t - 4, 0))$$



Example. Express the following function employing unit-step functions.

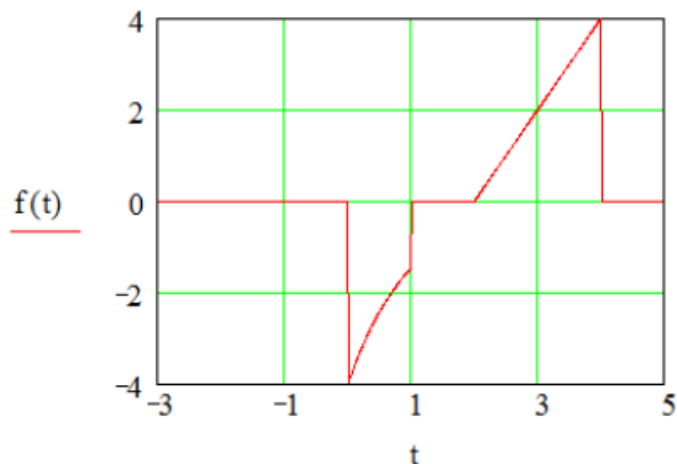
$$f(t) := \begin{cases} -4e^{-t} & \text{if } 0 \leq t < 1 \\ 2t - 4 & \text{if } 2 \leq t < 4 \\ 0 & \text{otherwise} \end{cases}$$

$$f(t) = -4e^{-t}[u(t) - u(t-1)] + (2t-4)[u(t-2) - u(t-4)]$$

Mathcad generated plot:

$$u(t) := \Phi(t)$$

$$f(t) := -4e^{-t} \cdot (u(t) - u(t-1)) + (2t-4) \cdot (u(t-2) - u(t-4))$$



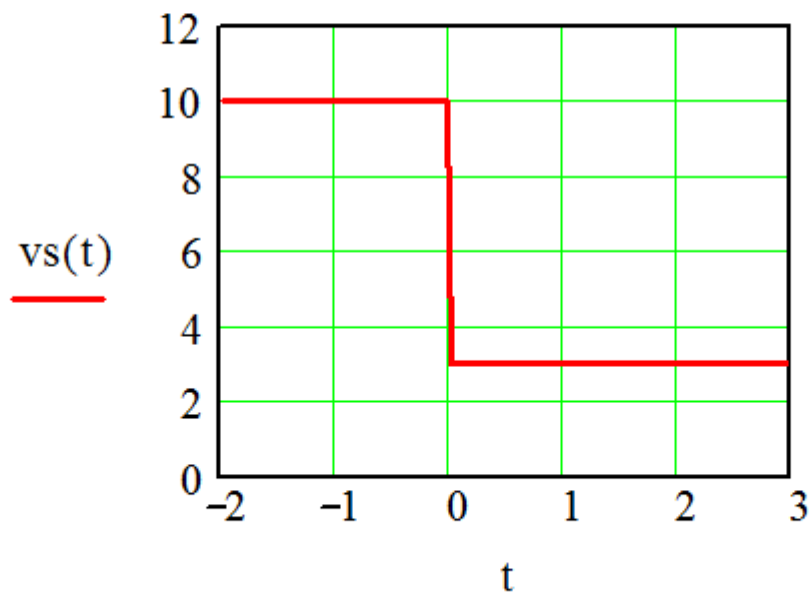
Employ Mathcad to recreate the plots on Slides 37 and 38 employing the three different representation of $f(t)$.

Submit as part of Assignment 8

Example: Plot the voltage source signal: $v_s(t) = 10 - 7u(t)$

$$u(t) := \Phi(t)$$

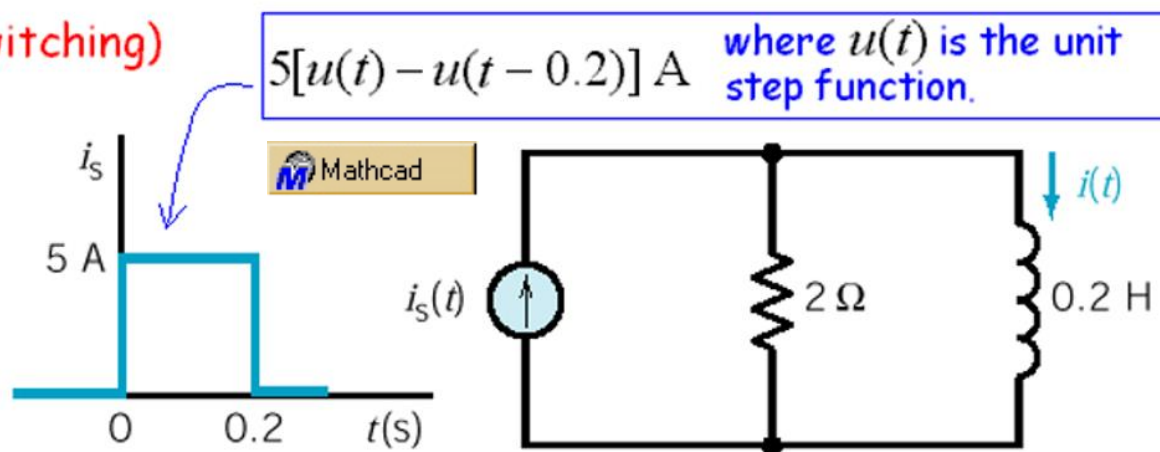
$$v_s(t) := 10 - 7u(t)$$



First-Order Circuit Analysis Examples

Example #4 (Multiple switching)

Find the inductor current. Assume that the initial inductor current is zero at $t = 0$.



This circuit behaves as if it had a switch in series with the current source that has been open for a long time and closes at $t = 0$. Then the switch closes for 0.2 sec and opens at $t = 0.2$ s.

Solution:

We first consider the first switching action. The complete response is (as was derived earlier)

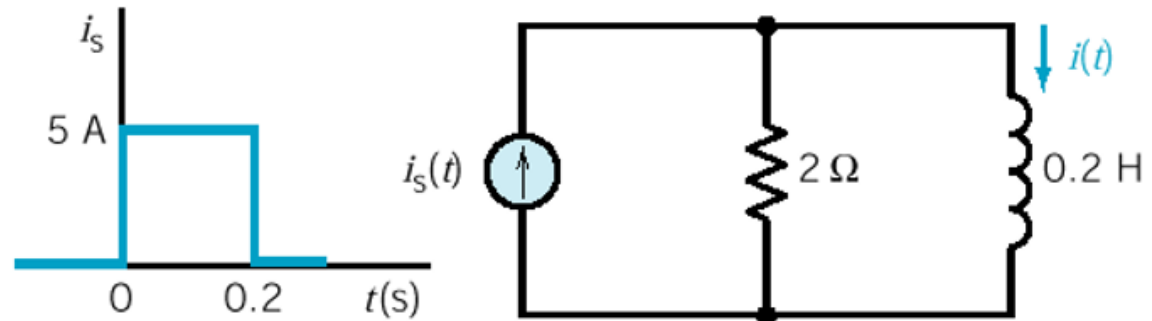
$$i(t) = [i(0) - I_s]e^{-\frac{R}{L}t} + I_s \quad \text{for } 0 < t < 0.2 \text{ s} \quad \left(\frac{R}{L} = 10\right)$$

$$i(t) = 5 - 5e^{-10t} \text{ A} \quad \text{for } 0 < t < 0.2 \text{ s}$$

First-Order Circuit Analysis Examples

Example #4 (Multiple switching)

Find the inductor current. Assume that the initial inductor current is zero at $t = 0$.



Solution: (cont.)

The solution for the current for $t > 0.2$ s is given by (with $I_s = 0$)

$$i(t) = [i(0.2) - I_s] e^{-\frac{R}{L}(t-0.2)} + I_s \quad \text{for } t > 0.2 \text{ s}$$

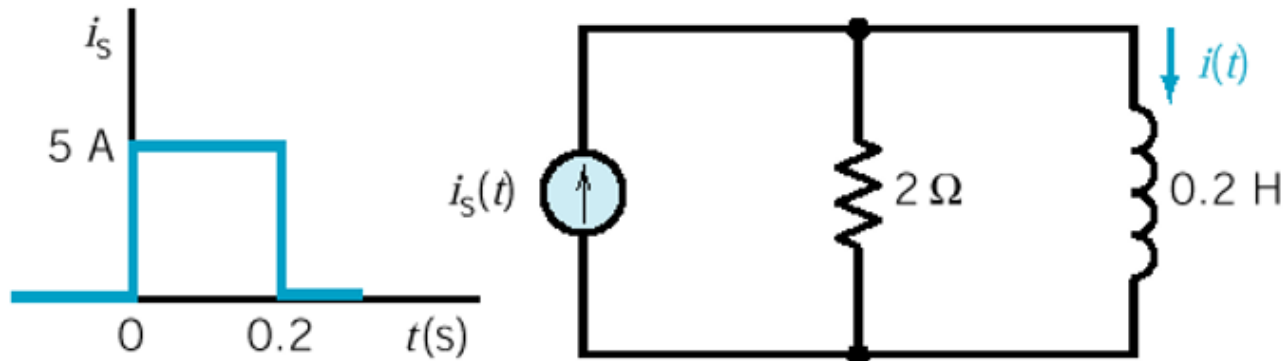
Note how the switching at t_0 other than zero changes the complete solution.

We need to know current at $t = 0.2$ s. This is the initial condition for the second switching operation. This is determined from the previous solution $i(t) = 5 - 5e^{-10t}$ A evaluated at $t = 0.2$ s

$$i(0.2) = 5 - 5e^{-(10)(0.2)} = 4.32 \text{ A. Therefore, } i(t) = [4.32 - 0]e^{-10(t-0.2)} + 0, \quad t > 0.2 \text{ s}$$

First-Order Circuit Analysis Examples

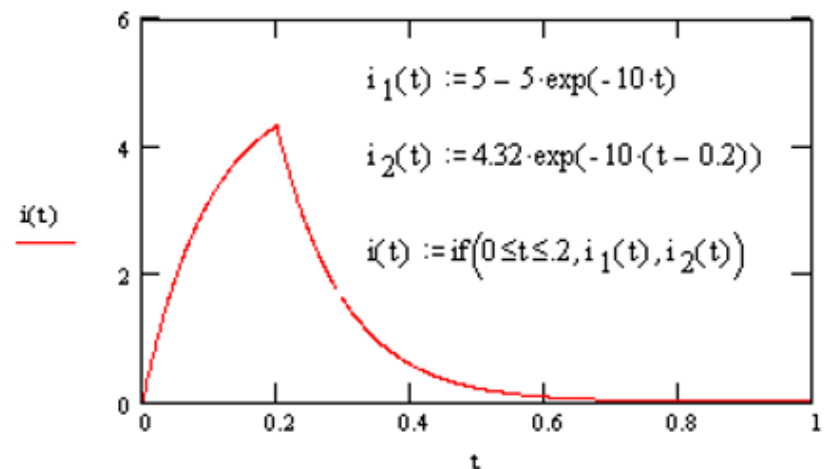
Example #4 (Multiple switching)



Summarizing:

$$i(t) = 5 - 5e^{-10t} \text{ A} \quad 0 < t < 0.2 \text{ s}$$

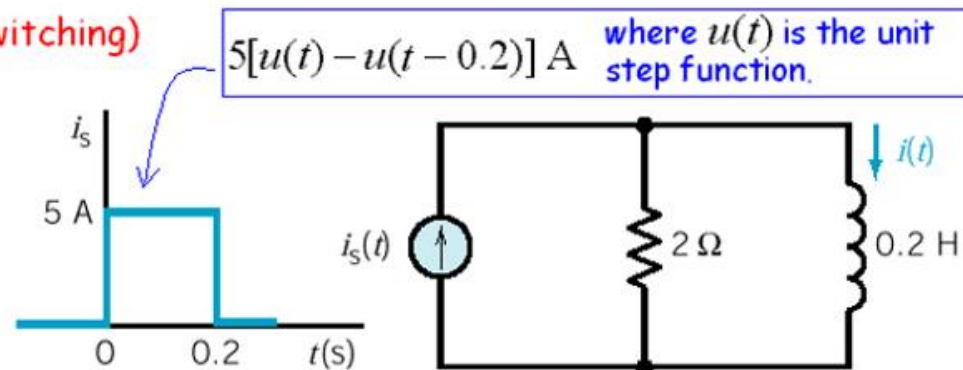
$$i(t) = 4.32e^{-10(t-0.2)} \text{ A} \quad t > 0.2 \text{ s}$$



Alternative Solution Employing Linear Circuit Properties

Example #4 (Multiple switching)

Find the inductor current. Assume that the initial inductor current is zero at $t = 0$.



The input current source can be expressed as

$$i_s(t) = i_{s1} + i_{s2} = 5u(t) - 5u(t - 0.2)$$

The inductor current response $i_1(t)$ for the input $i_{s1} = 5u(t)$ was obtained earlier:

$$i_1(t) = 5(1 - e^{-10t})u(t)$$

Employing the scaling and shift properties of linear circuits we can now express the response of the inductor current to the $i_{s2}(t) = -5u(t - 0.2)$ component as,

$$i_2(t) = -5(1 - e^{-10(t-0.2)})u(t - 0.2)$$

Alternative Solution Employing Linear Circuit Properties

Finally, by employing the superposition properties we can express the complete inductor current response as,

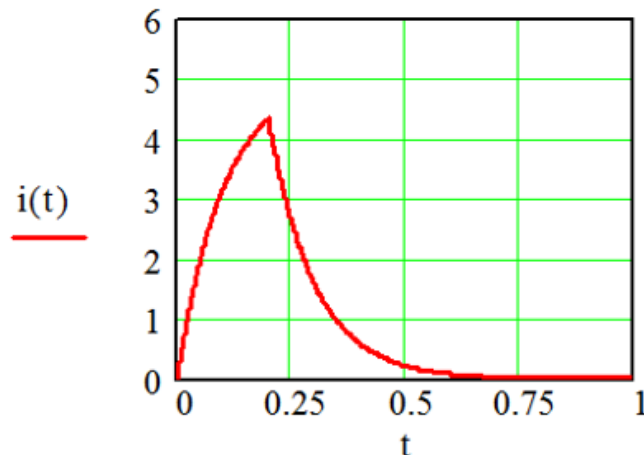
$$i(t) = i_1(t) + i_2(t) = 5(1 - e^{-10t})u(t) - 5(1 - e^{-10(t-0.2)})u(t - 0.2)$$

The unit-step terms are crucial and must be retained!

Mathcad plot of the the inductor current:

$$u(t) := \Phi(t)$$

$$i(t) := 5(1 - e^{-10t})u(t) - 5[1 - e^{-10(t-0.2)}]u(t - 0.2)$$



General Expression for the Complete Solution of First-Order Circuits Under Exponential Input

$$\frac{dx}{dt} + ax = f(t) \quad \text{with } f(t) = Ke^{-bt}, \quad b \neq a$$

As before, the solution is the sum of the homogeneous solution and the particular solution:

$$x = x_h + x_p$$

The homogeneous solution is independent of $f(t)$. It was determined earlier as

$$x_h = Ae^{-at}$$

The particular solution takes the form

$$x_p = Be^{-bt}$$

B is determined by substituting the particular solution in the above differential equation

$$-bBe^{-bt} + aBe^{-bt} = Ke^{-bt}$$

$$B(a - b) = K$$

$$B = \frac{K}{a - b} \quad b \neq a$$

General Expression for the Complete Solution of First-Order Circuits Under Exponential Input

$$\frac{dx}{dt} + ax = f(t) \quad \text{with } f(t) = Ke^{-bt}, \quad b \neq a$$

Now, we can express the complete solution of the above differential equation as

$$x(t) = Ae^{-at} + \frac{K}{a-b}e^{-bt}$$

If we know the initial condition $x(0)$, we can use it to solve for A

$$x(0) = A + \frac{K}{a-b}$$

or
$$A = \left(x(0) - \frac{K}{a-b} \right)$$

$$x(t) = \left(x(0) - \frac{K}{a-b} \right) e^{-at} + \frac{K}{a-b} e^{-bt}, \quad t > 0$$

Example: Complete Response of a Series RL Circuit to an Exponential Voltage Source

Find $i(t)$ for $t > 0$. Assume $i(0) = 2$ A.

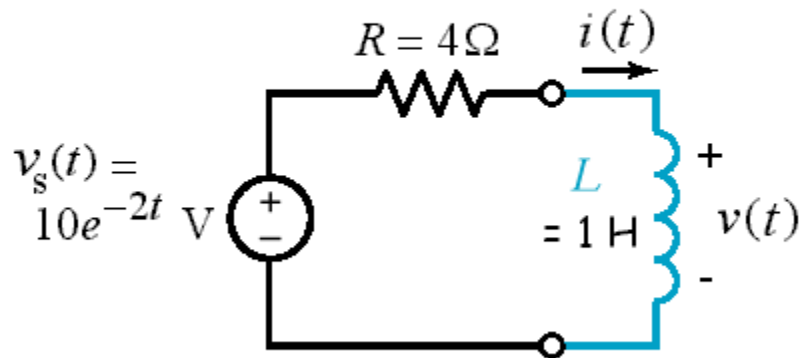
Mesh Analysis:

$$-v_s + Ri + v = 0 \quad \text{with } v = L \frac{di}{dt}$$

$$L \frac{di}{dt} + Ri = v_s$$

$$\frac{di}{dt} + \frac{R}{L}i = \frac{1}{L}v_s$$

$$\frac{di}{dt} + 4i = 10e^{-2t}, \quad t > 0$$



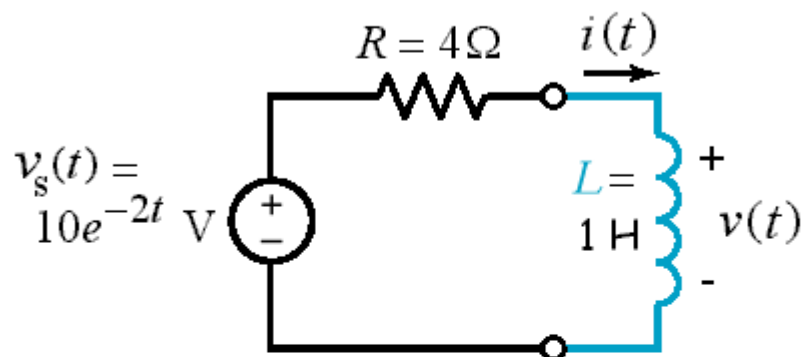
This last result is the differential equation describing this circuit. It has the general solution

$$x(t) = \left(x(0) - \frac{K}{a-b} \right) e^{-at} + \frac{K}{a-b} e^{-bt}, \quad t > 0$$

$$\text{where } x(t) = i(t) \quad x(0) = i(0) = 2 \text{ A} \quad K = 10 \quad a = 4 \quad b = 2$$

Example: Complete Response of a Series RL Circuit to an Exponential Voltage Source

Find $i(t)$ for $t > 0$. Assume $i(0) = 2$ A.



Solution (cont.):

Substituting the constants from the previous slide in the general solution form gives

$$i(t) = \left(2 - \frac{10}{4 - 2} \right) e^{-4t} + \frac{10}{4 - 2} e^{-2t}, \quad t > 0$$

or $i(t) = -3e^{-4t} + 5e^{-2t} \text{ A}, \quad t > 0$

General Expression for the Complete Solution of First-Order Circuits Under Exponential Input

$$\frac{dx}{dt} + ax = f(t) \quad \text{with} \quad f(t) = Ke^{-bt}, \quad \boxed{b = a}$$

As before, the solution is the sum of the homogeneous solution and the particular solution:

$$x = x_h + x_p$$

The homogeneous solution is independent of $f(t)$. It was determined earlier as

$$x_h = Ae^{-at}$$

The particular solution takes the form

$$x_p = Bte^{-bt} = Bte^{-at}$$

When the homogeneous solution already contains a term of the same form as the forcing function, we need to multiply the assumed form of the particular solution by t .

B is determined by substituting the particular solution in the above differential equation

$$-bBte^{-bt} + Be^{-bt} + aBte^{-bt} = Ke^{-bt}$$

$$Bte^{-bt}(a-b) + Be^{-bt} = Ke^{-bt}$$

$$\text{or } B = K \text{ since } b = a$$

General Expression for the Complete Solution of First-Order Circuits Under Exponential Input

$$\frac{dx}{dt} + ax = f(t) \quad \text{with } f(t) = Ke^{-bt}, \quad \boxed{b = a}$$

Therefore, $x_p = Kte^{-bt} = Kte^{-at}$

The complete solution can then be expressed as

$$x(t) = x_h + x_p = Ae^{-at} + Kte^{-at}$$

where the constant A is determined from the initial condition as

$$x(0) = Ae^{-0} + 0 = A$$

Finally, we get

$$\boxed{x(t) = x(0)e^{-at} + Kte^{-at}, \quad t > 0}$$

General Expression for the Complete Solution of First-Order Circuits Under Sinusoidal Input

$$\frac{dx}{dt} + ax = f(t) \quad \text{with} \quad f(t) = K \sin(\omega t)$$

Particular solution: $x_p = B_1 \sin(\omega t) + B_2 \cos(\omega t)$

We solve for the constants by substituting x_p in the differential equation, as follows

$$\omega B_1 \cos(\omega t) - \omega B_2 \sin(\omega t) + aB_1 \sin(\omega t) + aB_2 \cos(\omega t) = K \sin(\omega t)$$

$$\text{or } (aB_1 - \omega B_2) \sin(\omega t) + (\omega B_1 + aB_2) \cos(\omega t) = K \sin(\omega t) + (0) \cos(\omega t)$$

Now, by matching the coefficients of the sin and cos terms on both sides of the equation, we get

$$(\omega B_1 + aB_2) = 0$$

$$(aB_1 - \omega B_2) = K$$

Solving for B_1 and B_2 , we get

$$B_1 = \frac{aK}{a^2 + \omega^2} \quad B_2 = \frac{-\omega K}{a^2 + \omega^2}$$

General Expression for the Complete Solution of First-Order Circuits Under Sinusoidal Input

$$\frac{dx}{dt} + ax = f(t) \quad \text{with} \quad f(t) = K \sin(\omega t)$$

The complete solution can then be expressed as

$$x(t) = x_h + x_p = Ae^{-at} + B_1 \sin(\omega t) + B_2 \cos(\omega t)$$

where the constant A is determined from the initial condition as

$$x(0) = Ae^{-a \cdot 0} + B_1 \sin(\omega \cdot 0) + B_2 \cos(\omega \cdot 0) = A + B_2$$

or

$$A = [x(0) - B_2]$$

Finally, the solution is

$$x(t) = [x(0) - B_2]e^{-at} + B_1 \sin(\omega t) + B_2 \cos(\omega t), \quad t > 0$$

$$\text{where } B_1 = \frac{aK}{a^2 + \omega^2} \text{ and } B_2 = \frac{-\omega K}{a^2 + \omega^2}$$

General Expression for the Complete Solution of First-Order Circuits Under Sinusoidal Input

$$\frac{dx}{dt} + ax = f(t) \quad \text{with} \quad f(t) = K \cos(\omega t)$$

Note
cos term

By following a similar analysis to the preceding one, one can show that the complete solution is given by

$$x(t) = [x(0) - B_2]e^{-at} + B_1 \sin(\omega t) + B_2 \cos(\omega t), \quad t > 0$$

$$\text{where } B_1 = \frac{\omega K}{a^2 + \omega^2} \text{ and } B_2 = \frac{aK}{a^2 + \omega^2}$$

Summing Up

General Solution for the First-Order Differential Equation:

$$\frac{dx}{dt} + ax = f(t)$$

$f(t)$

K

$$x(t) = \left[x(0) - \frac{K}{a} \right] e^{-at} + \frac{K}{a}, \quad t > 0$$

$Ke^{-bt}, b \neq a$

$$x(t) = \left(x(0) - \frac{K}{a-b} \right) e^{-at} + \frac{K}{a-b} e^{-bt}, \quad t > 0$$

$Ke^{-bt}, b = a$

$$x(t) = x(0)e^{-at} + Kte^{-at}, \quad t > 0$$

$K \sin(\omega t)$

$K \cos(\omega t)$

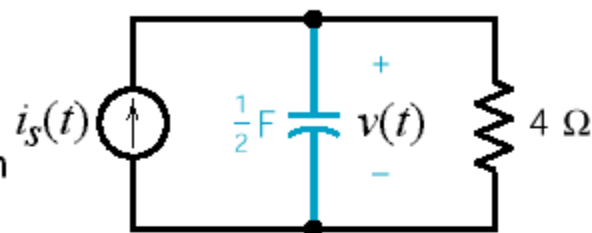
$$x(t) = [x(0) - B_2]e^{-at} + B_1 \sin(\omega t) + B_2 \cos(\omega t), \quad t > 0$$

where $B_1 = \frac{aK}{a^2 + \omega^2}$ and $B_2 = \frac{-\omega K}{a^2 + \omega^2}$

where $B_1 = \frac{\omega K}{a^2 + \omega^2}$ and $B_2 = \frac{aK}{a^2 + \omega^2}$

Response of First-Order Circuits to Sinusoidal Input

Example #1 Find the response $v(t)$ for the following circuit.
Assume steady-state operation for negative time.



Solution:

Since for $t < 0$ the current source is disconnected and circuit is in steady-state, then initial capacitor voltage is zero

$$v(0) = v(0^-) = 0$$

KCL at top node

$$C \frac{dv}{dt} + \frac{v}{R} = i_s$$

or

$$\frac{dv}{dt} + \frac{1}{2}v = 20\sin(2t), \quad t > 0$$

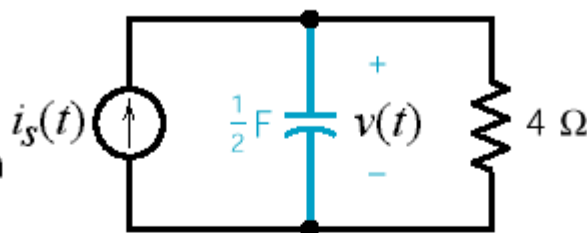
And we identify the constants:

$$a = 1/2, \quad w = 2, \quad K = 20$$

$$i_s(t) = [10\sin(2t)]u(t) \text{ A}$$
$$= \begin{cases} 0 & t < 0 \\ 10\sin(2t) & t \geq 0 \end{cases}$$

Response of First-Order Circuits to Sinusoidal Input

Example #1 Find the response $v(t)$ for the following circuit.
Assume steady-state operation for negative time.



Solution: (cont.)

The complete solution is determined from the previous table as

$$x(t) = [x(0) - B_2]e^{-at} + B_1 \sin(\omega t) + B_2 \cos(\omega t), \quad t > 0$$

$$\text{where } B_1 = \frac{aK}{a^2 + \omega^2} \text{ and } B_2 = \frac{-\omega K}{a^2 + \omega^2}$$

Hence,

$$v(t) = \left(0 - \frac{(2)(20)}{1/4 + 4} \right) e^{-t/2} + \frac{(1/2)(20)}{1/4 + 4} \sin(2t) + \frac{-(2)(20)}{1/4 + 4} \cos(2t) \text{ V}$$

$$v(t) = \frac{160}{17} e^{-t/2} + \frac{40}{17} \sin(2t) - \frac{160}{17} \cos(2t) \text{ V}, \quad t > 0$$

Response of First-Order Circuits to Sinusoidal Input

Plotting voltage Signal Using Mathcad

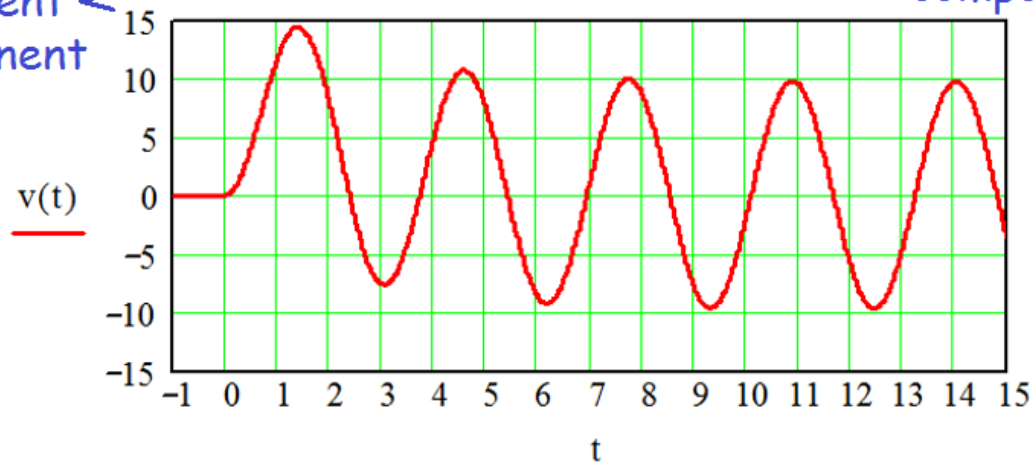
$$v(t) = \frac{160}{17}e^{-t/2} + \frac{40}{17}\sin(2t) - \frac{160}{17}\cos(2t) \text{ V, } t > 0$$

$$u(t) := \Phi(t)$$

$$v(t) := \left(\frac{160}{17} \cdot e^{-\frac{1}{2}t} + \frac{40}{17} \cdot \sin(2 \cdot t) - \frac{160}{17} \cdot \cos(2 \cdot t) \right) u(t)$$

Transient
component

Steady-state
component



Verification using Matlab's Symbolic Toolbox solution.

$$\frac{dv}{dt} + \frac{1}{2}v = 20\sin(2t), \quad v(0) = 0$$

Matlab Commands:

```
>> v=dsolve('Dv+0.5*v=20*sin(2*t)','v(0)=0');  
>> simple(v)  
  
-160/17*cos(2*t)+40/17*sin(2*t)+160/17*exp(-1/2*t)
```

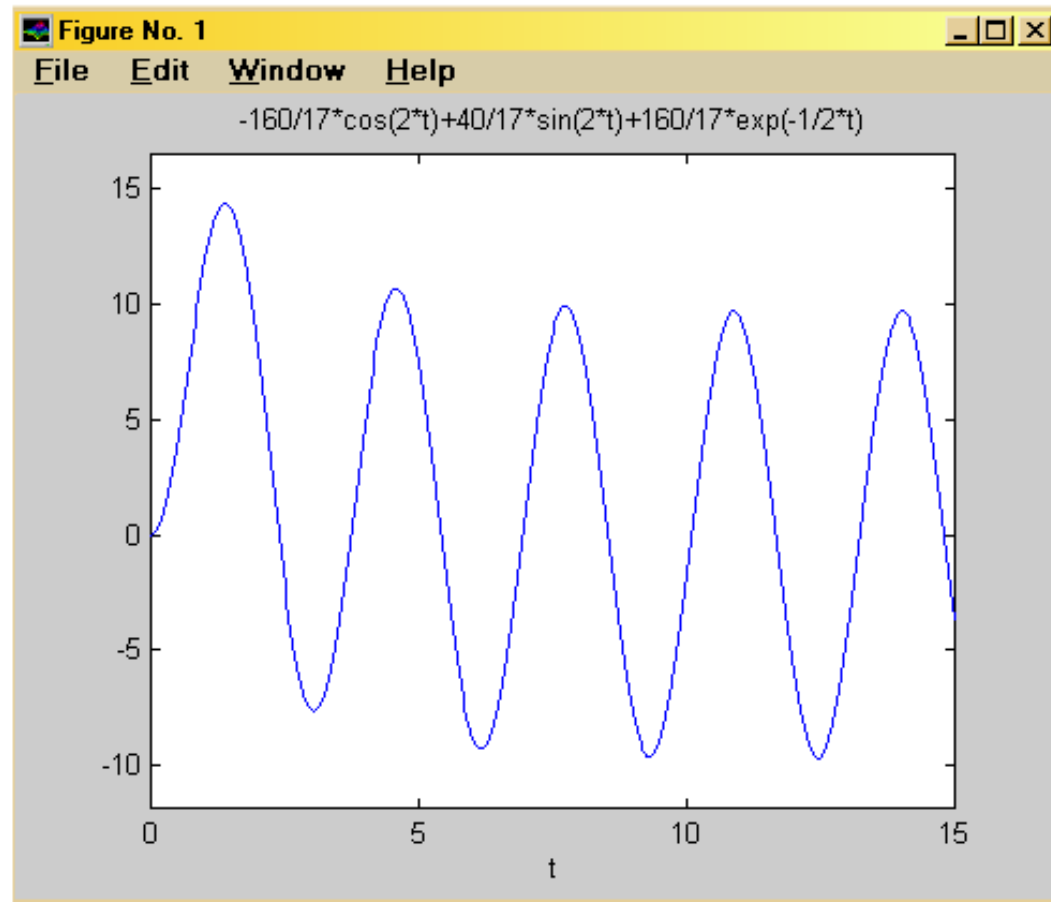
Compare ans. to:

$$v(t) = \frac{160}{17}e^{-t/2} + \frac{40}{17}\sin(2t) - \frac{160}{17}\cos(2t) \quad \text{V, } t > 0$$

Verification using Matlab's Symbolic Toolbox solution.

Matlab Commands:

```
» v=dsolve('Dv+0.5*v=20*sin(2*t)','v(0)=0');  
» simple(v)  
  
-160/17*cos(2*t)+40/17*sin(2*t)+160/17*exp(-1/2*t)  
» ezplot(v,[0,15])
```



Ti89 Calculator ODE Solution Verification

$$v = \frac{-40 \cdot e^{-\frac{t}{2}} \cdot \left(4 \cdot e^{\frac{t}{2}} \cdot \cos(2 \cdot t) - e^{\frac{t}{2}} \cdot \sin(2 \cdot t) - 4 \right)}{17}$$

`desolve(v' + (1/2)v = 20sin(2t) and v(0)=0, t, v)`

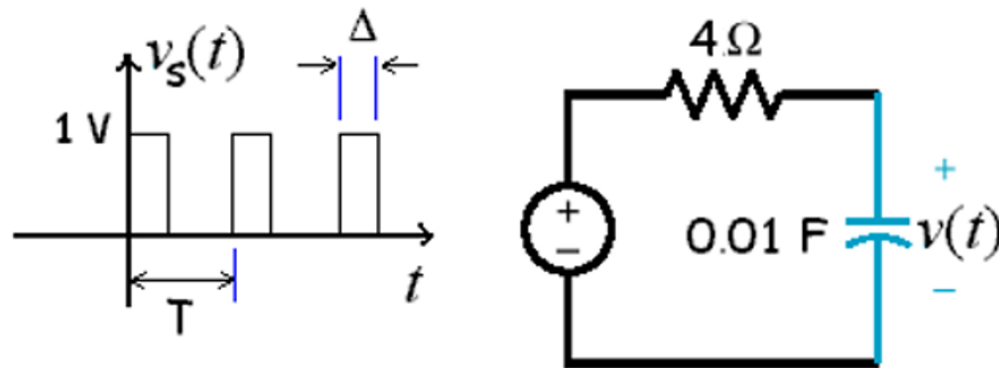
Which can be put in the form we obtained earlier:

$$v(t) := \left(\frac{160}{17} \cdot e^{-\frac{1}{2}t} + \frac{40}{17} \cdot \sin(2 \cdot t) - \frac{160}{17} \cdot \cos(2 \cdot t) \right)$$

Example

Employ Multisim to solve for the capacitor voltage

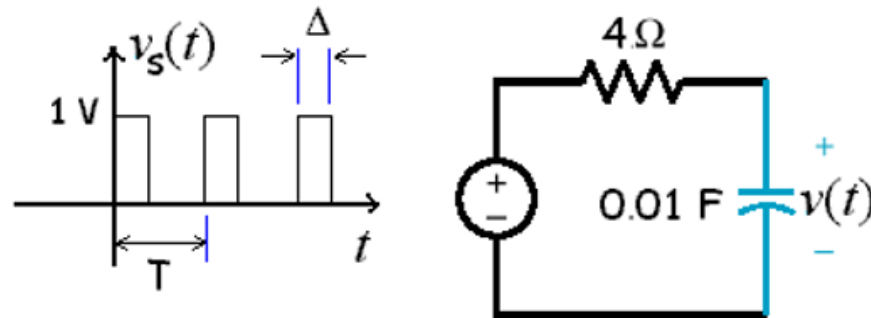
Assume the voltage source to be a train of small duration 1 V pulses with a frequency of 2 Hz (i.e., period of 0.5 sec). Experiment with different pulse widths.



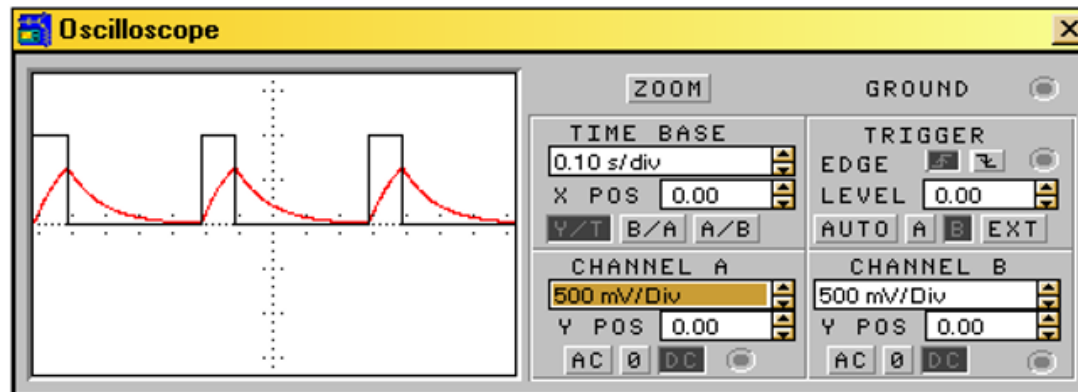
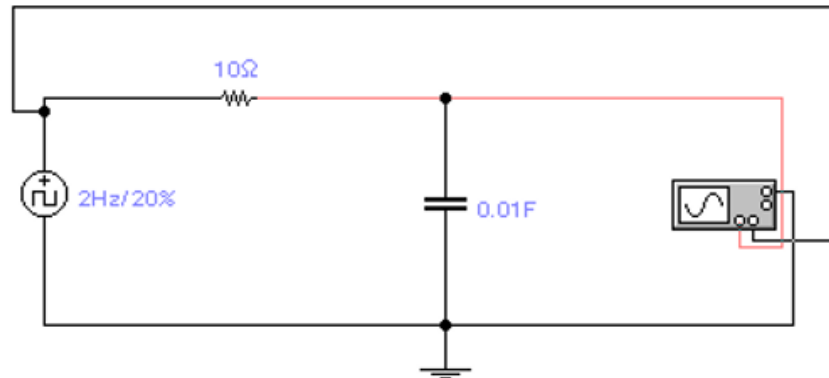
$$\text{Period } T = 1/\text{freq} = 1/2\text{Hz} = 0.5\text{ s}$$

$$\text{Pulse width: } \Delta \quad (0 < \Delta < T)$$

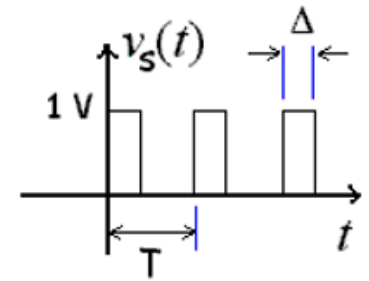
Example (cont.)



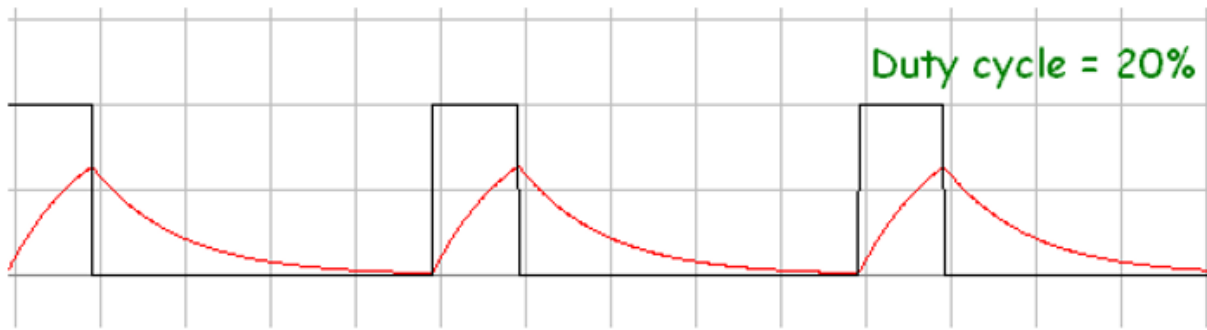
Electronics Workbench



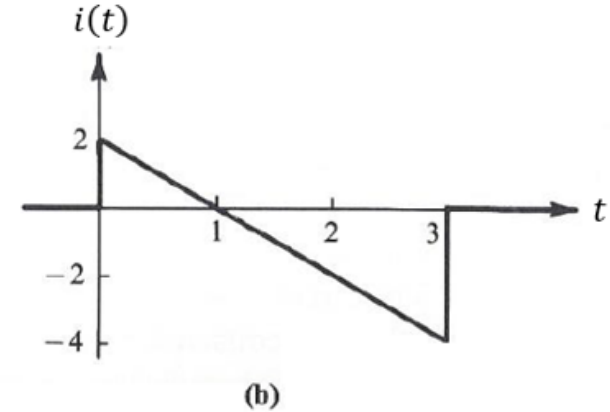
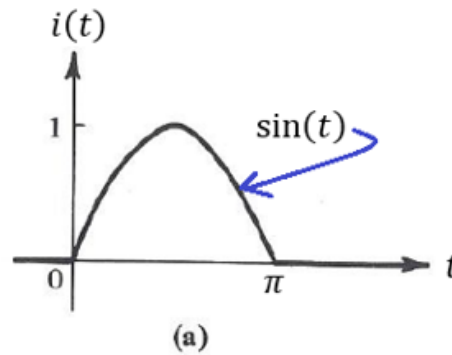
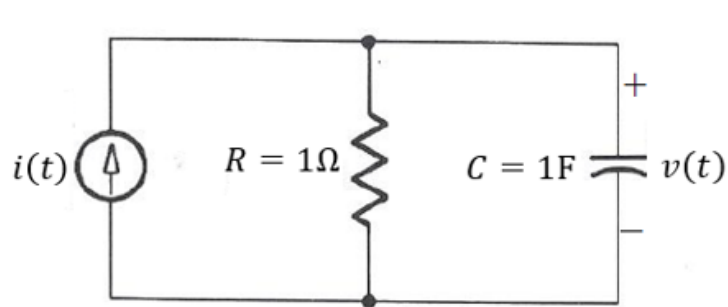
Example (cont.)



$$\text{Duty Cycle} = \frac{\Delta}{T}$$

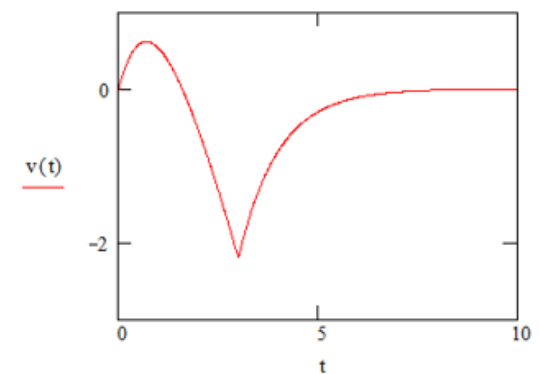
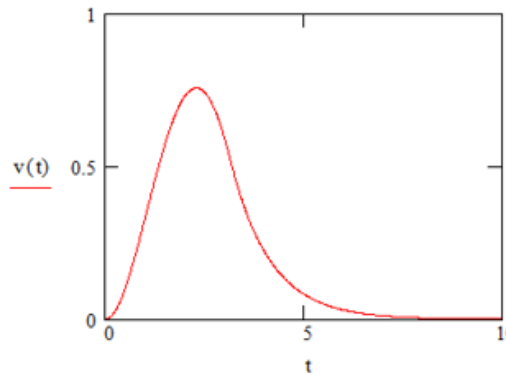


Your turn: Solve for the capacitor voltage when the input current source has the waveform shown in (a). Repeat for the waveform in (b). Assume $v(0) = 0$. Plot the responses.



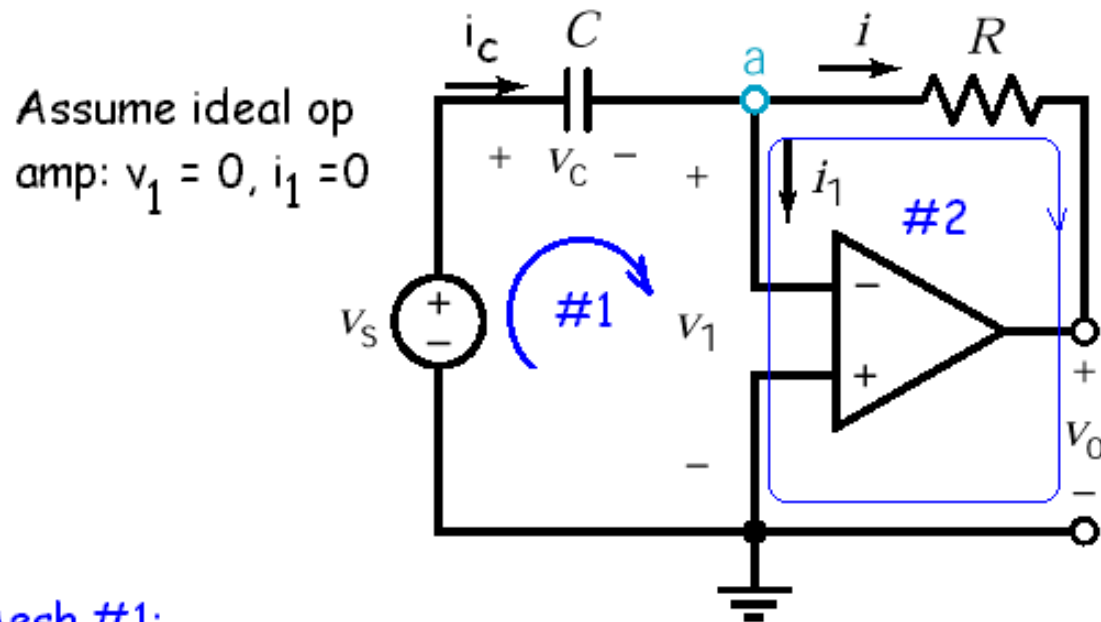
Hint (Part a): $i(t) = \sin(t)u(t) + \sin(t - \pi)u(t - \pi)$
Employ shift and superposition properties

Ans. (plots)



The Operational Amplifier and RC Circuits

The Differentiator Circuit



Let us solve for the output voltage as a function of the input voltage.

$$v_o = -RC \frac{dv_s}{dt}$$

Mesh #1:

$$-v_s + v_c = 0 \quad \text{or} \quad v_c = v_s$$

Mesh #2:

$$Ri + v_o = 0 \quad \text{or} \quad v_o = -Ri$$

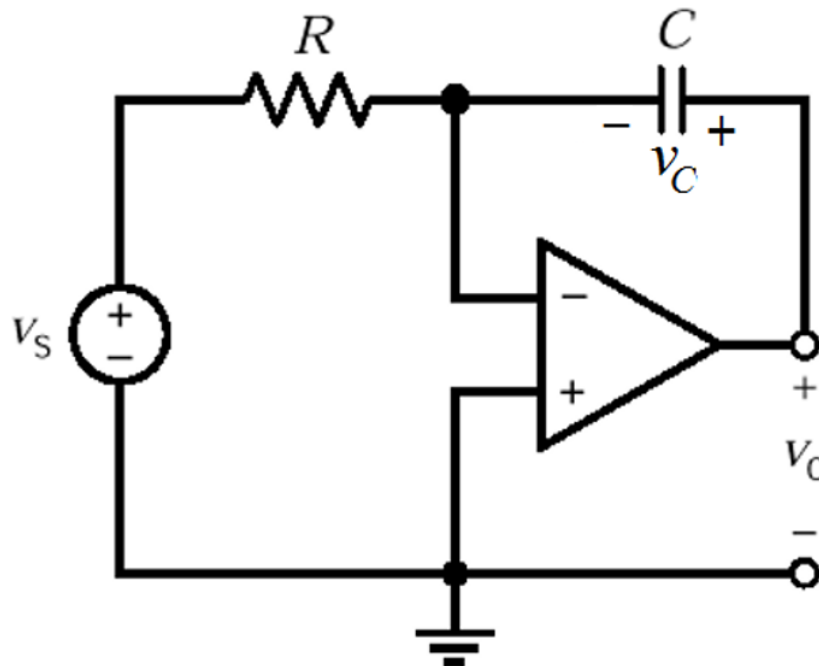
KCL at node a:

$$i + i_1 = i_c \quad \text{or} \quad i = i_c$$

$$\begin{aligned} \text{Therefore, } v_o &= -Ri_c = -RC \frac{dv_c}{dt} \\ \text{or } v_o &= -RC \frac{dv_s}{dt} \quad (\text{since } v_c = v_s) \end{aligned}$$

The Operational Amplifier and RC Circuits

The Integrator Circuit



$$v_o = -\frac{1}{RC} \int_0^t v_s d\tau + v_o(0)$$

Your turn: Derive the integrator input/output relationship shown

If the circuit starts operation with zero charge on its capacitor, then $v_o(0) = 0$

$$v_o(0) = v_c(0)$$

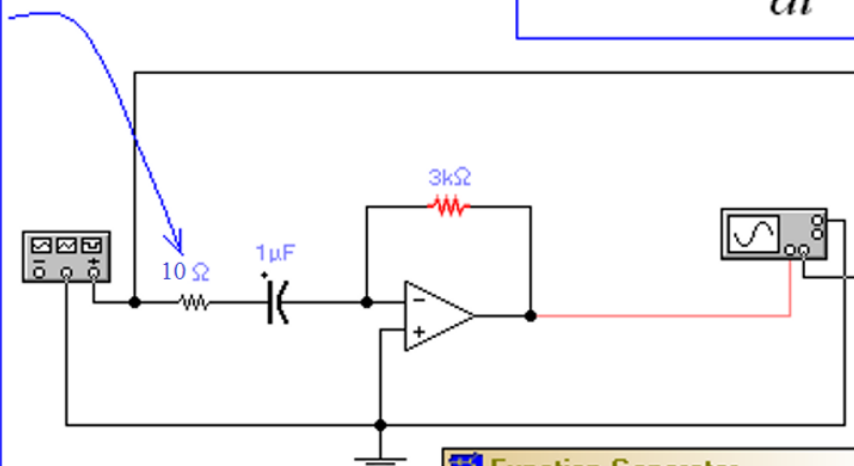
The Operational Amplifier and RC Circuits

Workbench Example Differentiator Circuit

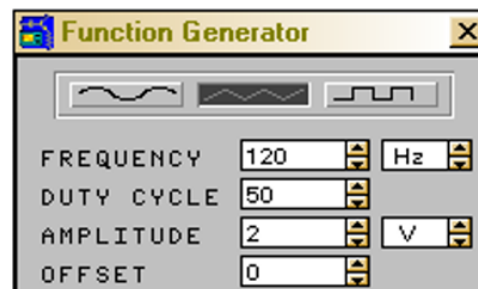
This small resistor is used to eliminate damped oscillations in the output signal at rising edge.

Note: The negative sign in the formula causes the differentiated signal to flip upside down.

$$v_o = -RC \frac{dv_s}{dt}$$

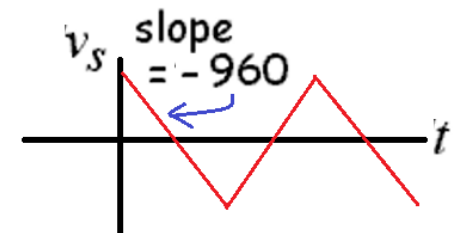


Electronics Workbench

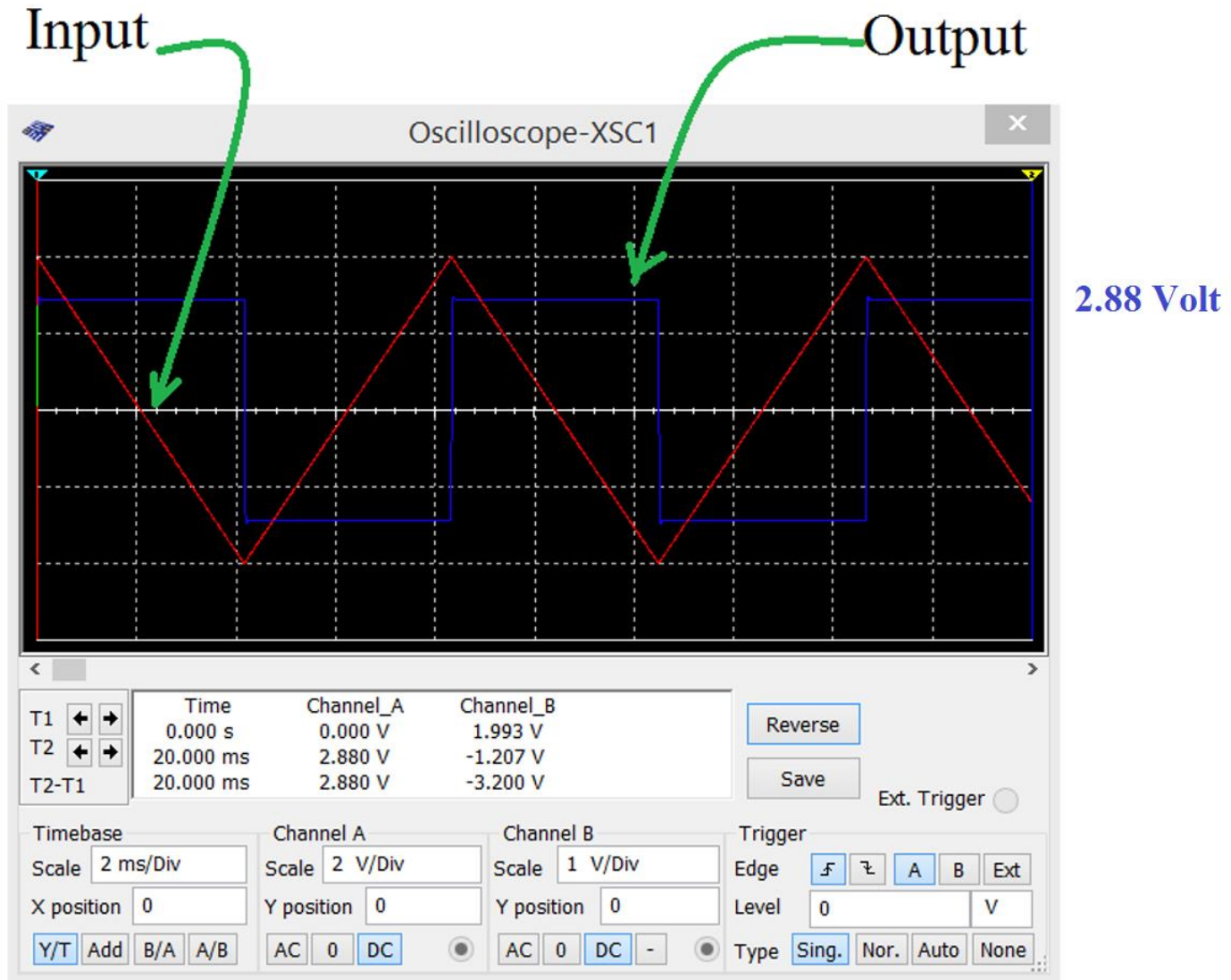


Verification:

120 Hz implies a 0.00833 s period. The slope of the linear rising edge of the 2 V amplitude input signal is: 2(amplitude)/(half period) or slope = - 4/0.00417 = - 960 V/s. Now, $-(-960)(RC) = 960(0.003) = 2.88$ V which is the output signal amplitude displayed on the scope.

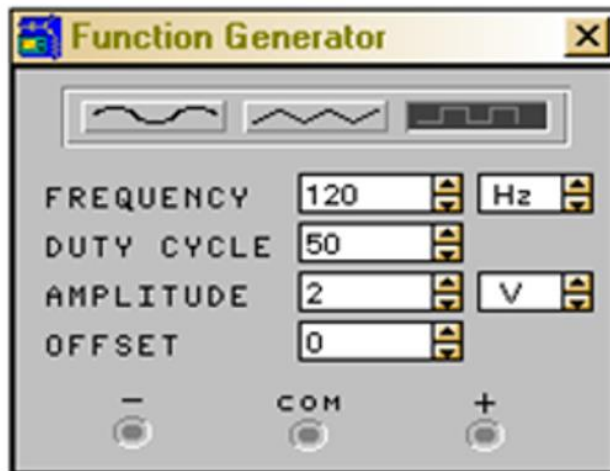


Differentiator (cont.)



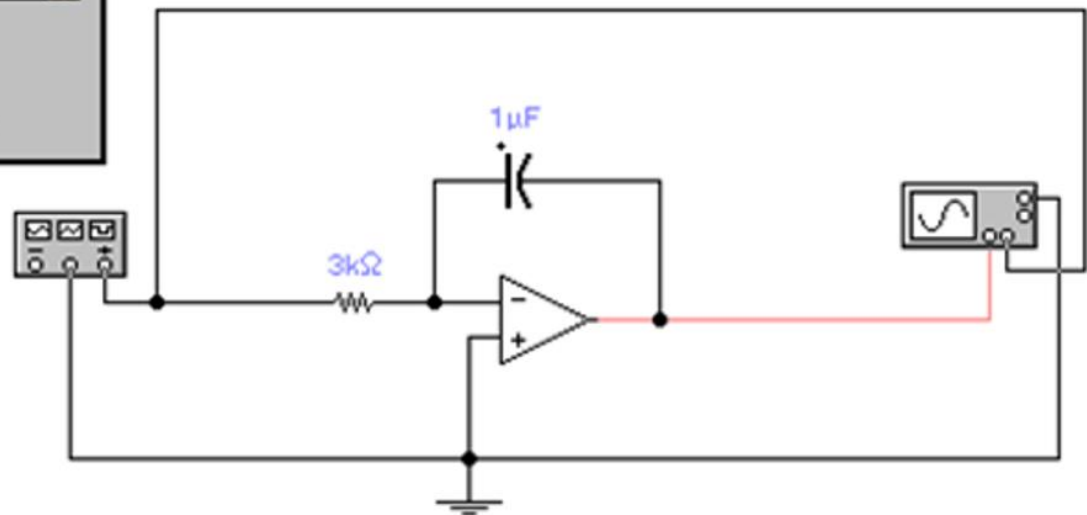
Integrator Circuit

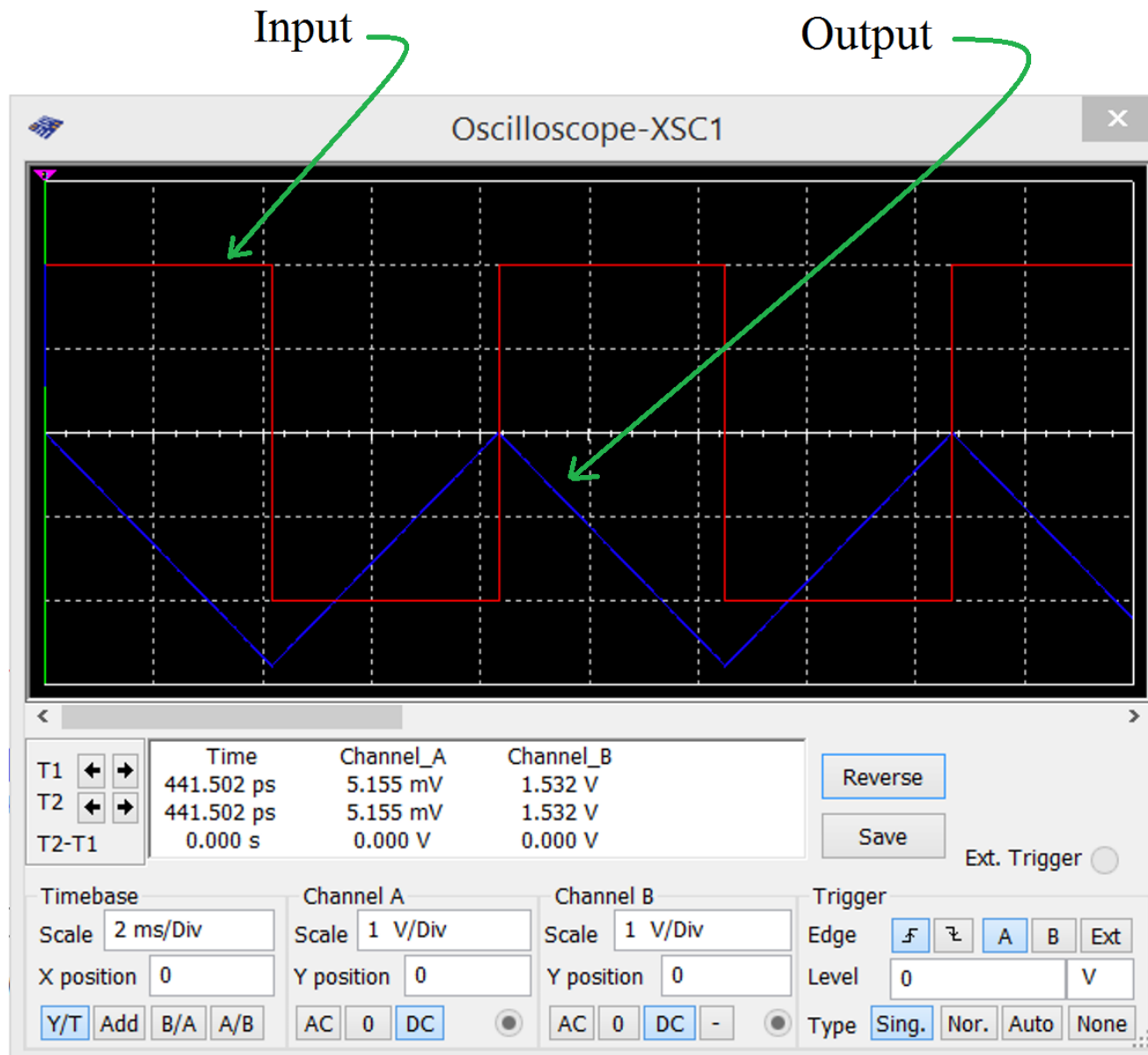
Demo



$$v_o = -\frac{1}{RC} \int_0^t v_s d\tau + v_o(0)$$

$$v_o(0) = v_c(0) = 0$$





Integrator Circuit

Example

Determine and plot the output of the integrator circuit (for $t \geq 0$) to the input voltage signal $v_s(t) = 2\sin(240\pi t)$. Assume $v_o(0) = 0$, $R = 3\text{k}\Omega$ and $C = 1\mu\text{F}$. (Note: The signal has a radian frequency of $\omega = 240\pi$ which corresponds to a Hz frequency of $f = \frac{\omega}{2\pi} = 120\text{Hz}$)

Solution:

$$\begin{aligned}v_o &= -\frac{1}{RC} \int_0^t v_s(\tau) d\tau + v_o(0) \\&= -\frac{1}{(3000)10^{-6}} \int_0^t 2 \sin(240\pi\tau) d\tau + 0 \\&= -\frac{1000}{3} \left(\frac{-2}{240\pi} \cos(240\pi\tau) \Big|_0^t \right) \cong 0.884 \cos(240\pi\tau) \Big|_0^t \\&= 0.884 \cos(240\pi t) - 0.884\end{aligned}$$

Repeat with
 $v_s(t) = 2\cos(240\pi t)$

Function Generato...

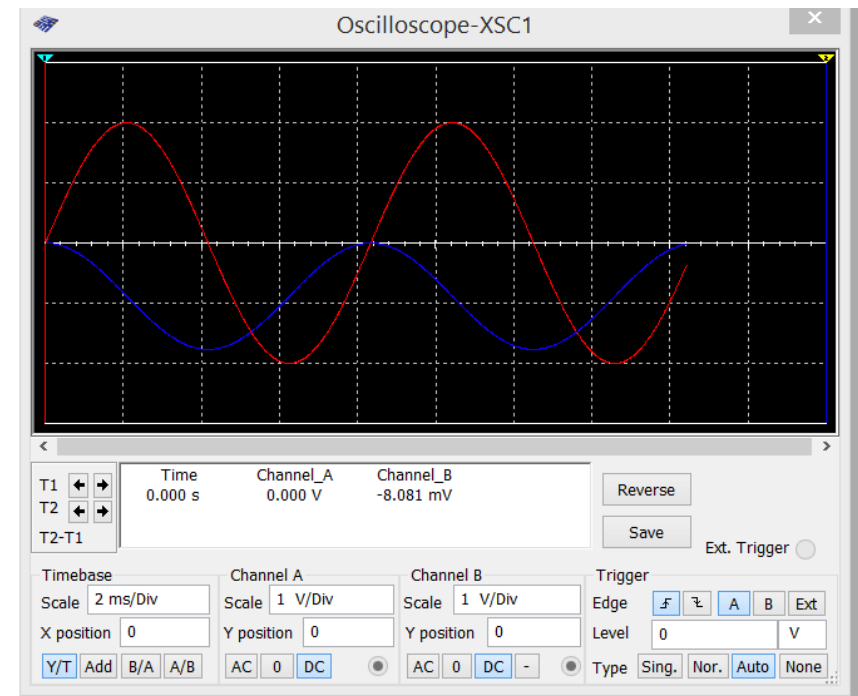
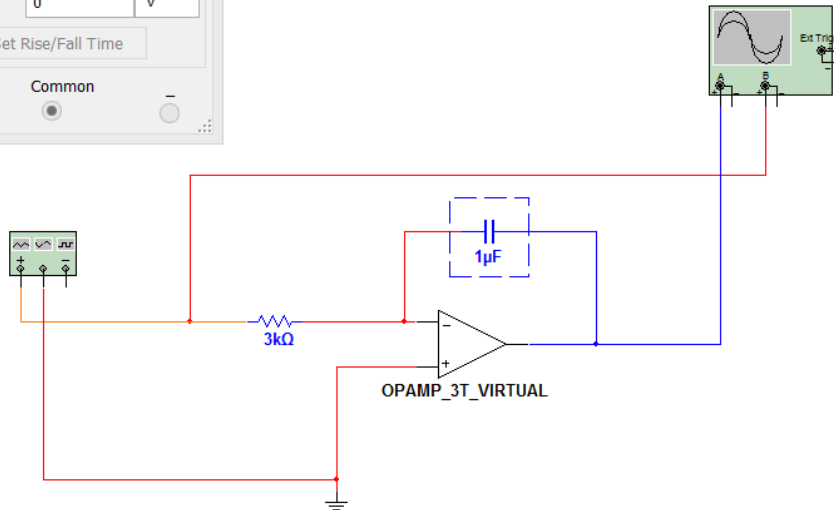
Waveforms

Signal Options

Frequency	120	Hz
Duty Cycle	50	%
Amplitude	2	Vp
Offset	0	V

Set Rise/Fall Time

Common



Capacitor

Label Display Value Fault Pins User Fields

Capacitance (C): 1μ F

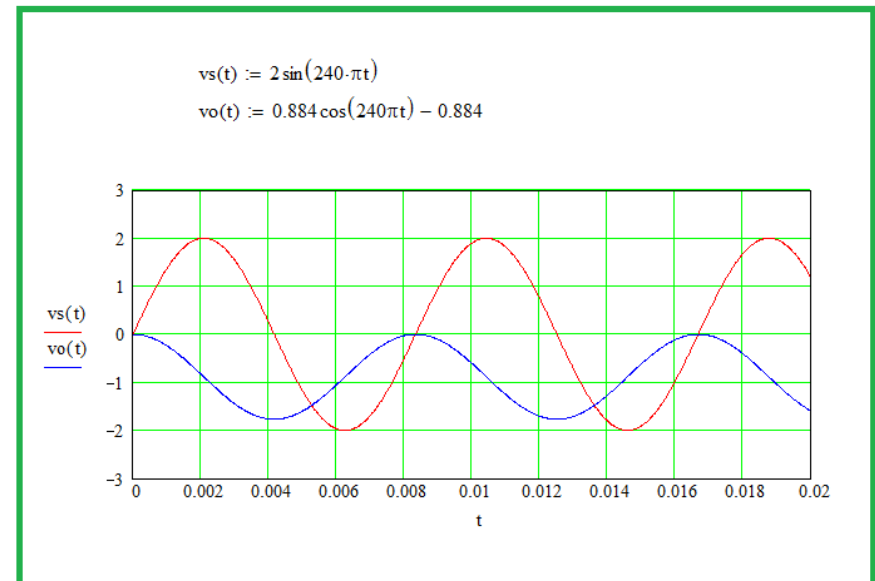
Tolerance: 0 %

Component Type:

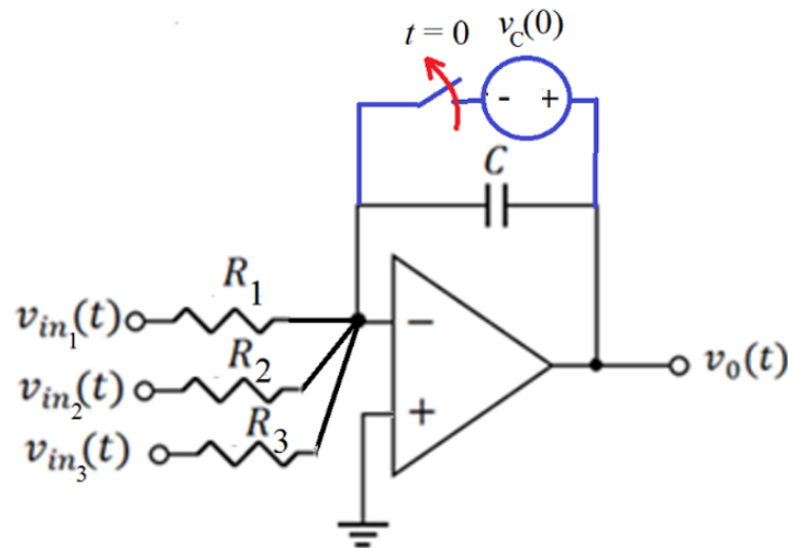
Hyperlink:

Additional SPICE Simulation Parameters

☒ Initial Conditions: 0 V



A generalization of the above basic integrator is the inverting summing integrator shown below. This circuit is the basis for the analog computers of the 50's and 60's era.



The above circuit has the following input/output relationship,

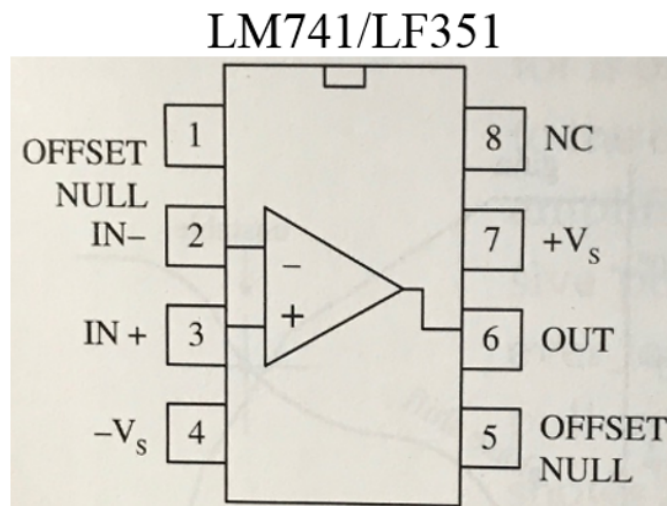
$$v_o(t) = - \int_{0^-}^t \left[\frac{1}{R_1 C} v_{in1}(\tau) + \frac{1}{R_2 C} v_{in2}(\tau) + \frac{1}{R_3 C} v_{in3}(\tau) \right] d\tau + v_c(0^-)$$

$$v_o(0^+) = v_c(0^+) = v_c(0^-)$$

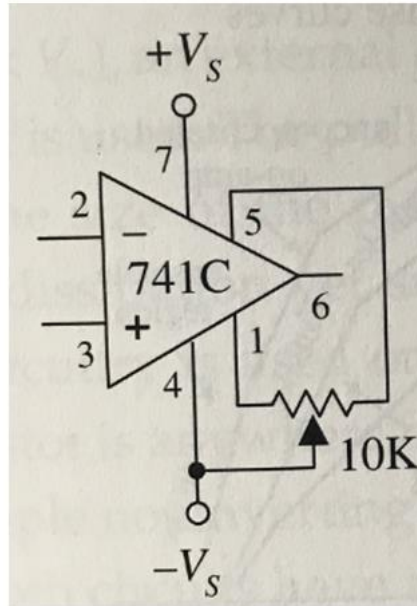
Practical Op-Amp Integrator Considerations:

Input Voltage Offset, Input Current Offset and DC Compensation

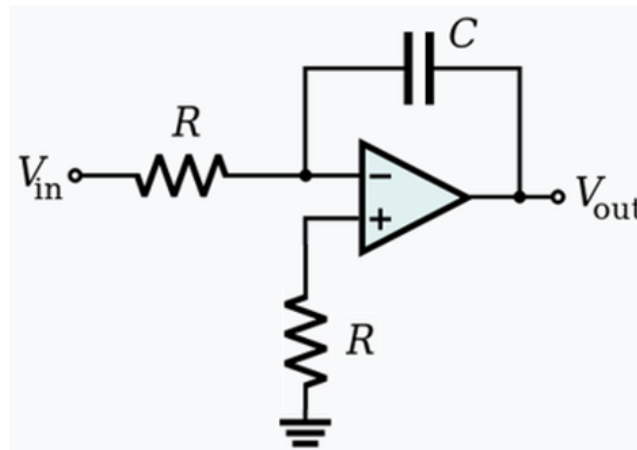
Certain modifications to the above integrator circuit are required in practice since the op amp is not a perfect (ideal) device. For example, the output of a practical op amp (say the LM741) saturates to a value close to one of its power supply voltages, even when both of its input terminals are grounded. This is because the voltage difference between the +ve and -ve terminals of the op amp is not a perfect zero. Practical op amps exhibit a small input offset voltage (about 1mV for the LM741 and 5mV for the wideband LF 351) that causes the output to saturate. This offset voltage is problematic for the integrator circuit: It results in a drift (cumulative summing) of the output voltage. This problem may be overcome by zeroing the input offset voltage. Practical op amp ICs include a pair of voltage offset nulling terminals. These would be pins 1 and 5 for the LM741 and LF351 as depicted in the figure below.



A $10\text{k}\Omega$ potentiometer is placed between these two terminals, while the pot's wiper (center pin) is connected to the negative supply voltage ($-V_S$), as shown in the figure below. The procedure to minimize the offset voltage is to short the op amp input terminals (pin 2 and 3) to ground and then adjust the pot until the output of the op amp approaches zero volts (as monitored on a voltmeter).



A second problem with some practical bipolar op amps (e.g., LM741) is the relatively large input bias currents (on the order of 80nA). These are the currents that flow into the input terminals of the op amp (ideally, they were assumed to be zero). The presence of the input bias current leads to output voltage errors in amplifiers and integrators. One solution would be to add a resistor R between the non-inverting input and ground as shown in the figure below. Generally speaking, this resistor is set to a value that is equal to the parallel combination of all resistors connected to the inverting input of the op amp. For the basic integrator circuit, the value of this resistor R is selected equal to that of the input resistor.

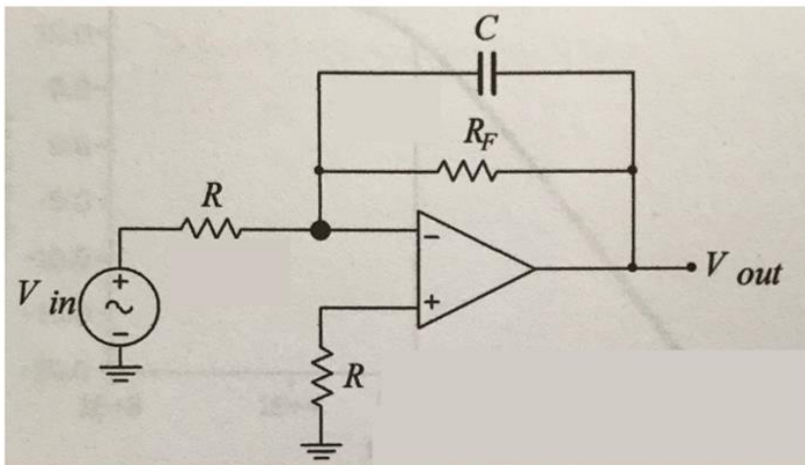


On the other hand, the JFET LF351 op amp has very, very small input bias currents (on the order of 50pA) and there is no need for the above compensation resistor (but it does not hurt if it is used).

An alternative solution to the input voltage offset and input current bias problems is to use (more expensive) precision op amps, such as the OPA277. This precision op amp has a very small input bias current (on the order of 0.5nA) and an ultra-low input offset voltage (on the order of $\pm 10\mu\text{V}$). For such precision op amps, there is usually no need for the above compensation circuitry.

Integrator Practical Considerations (Cont.)

There is also one more problem with practical integrator circuits that one should pay close attention to. It is possible for a tiny DC offset voltage to be present in the input signal. This offset results in drift (cumulative summing) of the output voltage. Small DC offsets (V_{DC}) lead to steady accumulation of charge in the capacitor until the op amp output saturates. One practical solution for eliminating output voltage drift is to provide another feedback path for DC through a high valued resistor (say, $R_F \gg R$) in parallel with the capacitor as shown in the following figure.

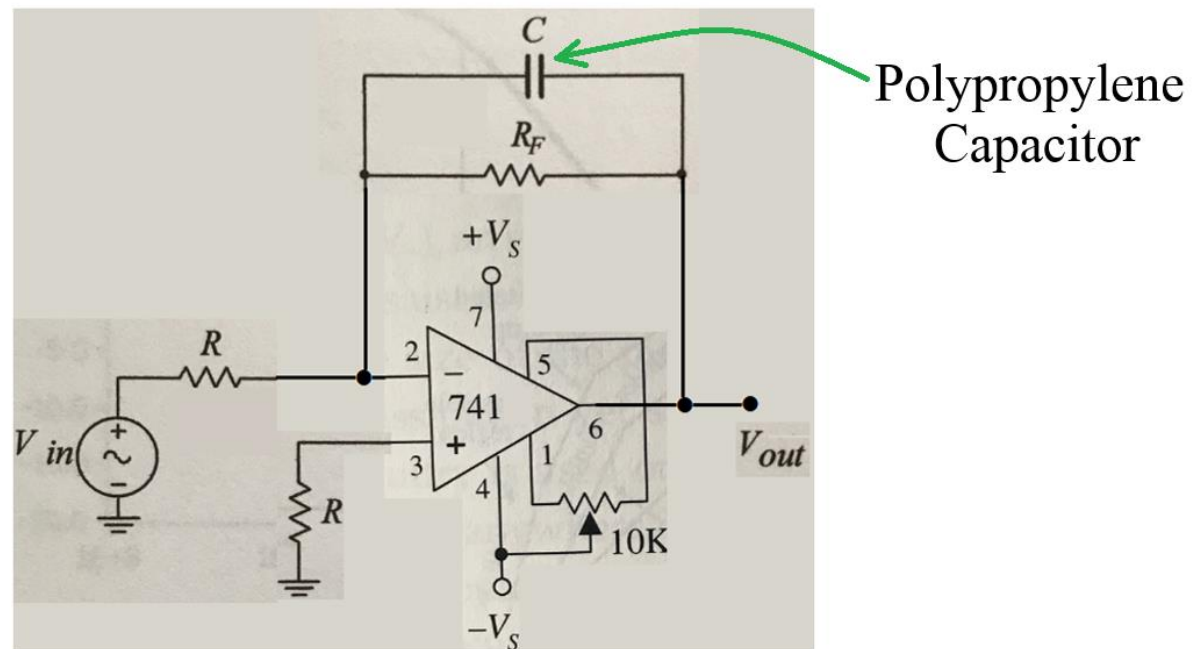


Your turn: Solve for $v_o(t)$, $t > 0$ with and without R_F . Assume:
 $v_o(0) = 0$, $C = 0.1\mu\text{F}$, and
 $v_{in}(t) = \cos(2\pi t) + 10^{-4}$
What happens to $v_o(t)$ as t becomes large.

This feedback resistor will hold the DC component contribution to the output voltage (through the process of negative feedback) at a value of $-\frac{R_F}{R} V_{DC}$. For example, for $V_{DC} = 0.1\text{mV}$, $R_F = 1\text{M}\Omega$ and $R = 10\text{k}\Omega$ the DC component contribution to the output voltage is controlled at -0.01 Volt . The addition of resistor R_F to the circuit should not affect the integrated signal as long as the frequency of the input signal satisfies, $f > \frac{1}{2\pi C R_F}$.

The dielectric absorption (DA) coefficient of a capacitor will also cause drift. A capacitor DA significantly contributes to an op amp's overall settling time; a capacitor with high DA does not completely discharge, and the residual voltage leads to op amp output drift. This problem can be minimized if a capacitor with low DA, such as polypropylene, polystyrene and mica, is used. Electrolytic and tantalum capacitors exhibit high DA and must not be used in op amp based integrators.

In summary, the following figure depicts the complete circuit for a practical inverting integrator, employing the readily available LM741 (or LF351) op amp.

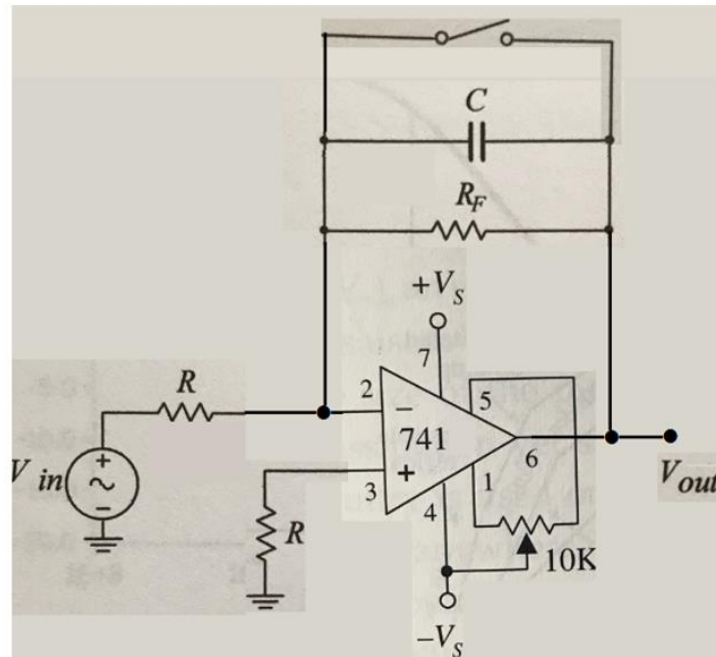


The initial condition $v_o(0)$ can be set by charging the capacitor to that value. If the integrator is to be reset to $v_o(0) = 0$, we would simply briefly short circuit (discharge) the capacitor.

Experimental results for a *practical* integrator employing the *imprecise* LM741 operational amplifier

Element values: $R_F = 1\text{M}\Omega$, $R = 10\text{k}\Omega$, $C = 0.1\mu\text{F}$ (Polypropylene) and supply voltages $\pm V_s = \pm 15\text{V}$. The input signal is a 2V_{pp} 159Hz (or 1000 rad/sec) periodic signal (sine, square, or triangular) with zero offset and 50% duty cycle.

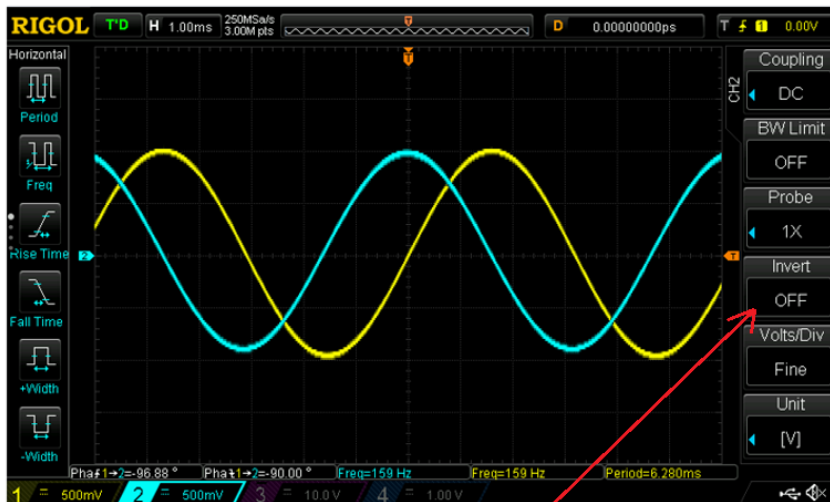
(The 10K pot is adjusted to eliminate the dc component in the output signal)



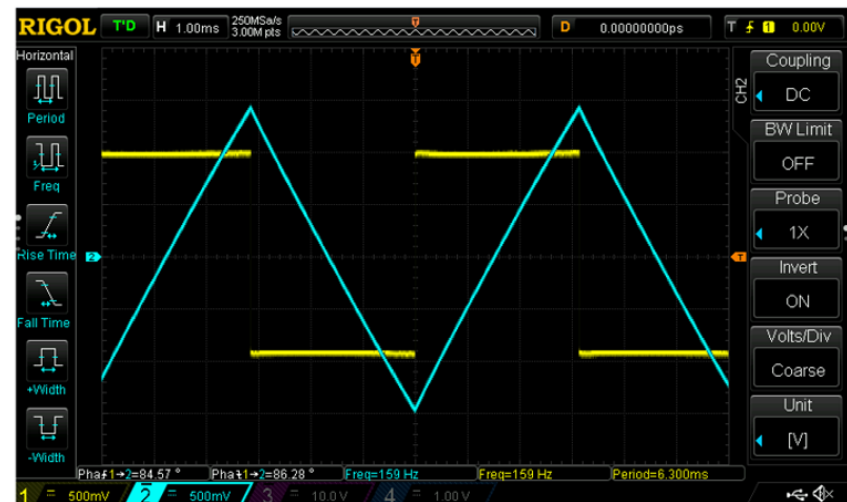
The switch is used to set the initial output to zero.

(For better visualization, the scope inversion is switched ON on CH2 so that the integrator output is multiplied by -1)

Sine wave input

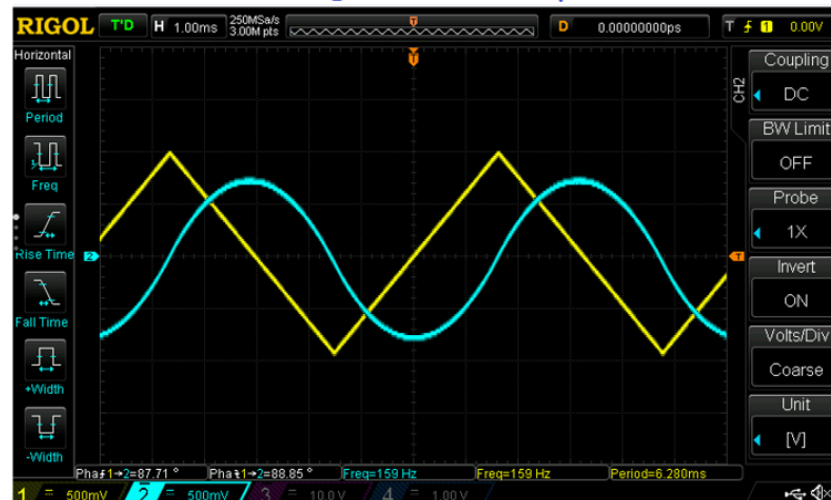


Square wave input



Triangular wave input

Invert OFF

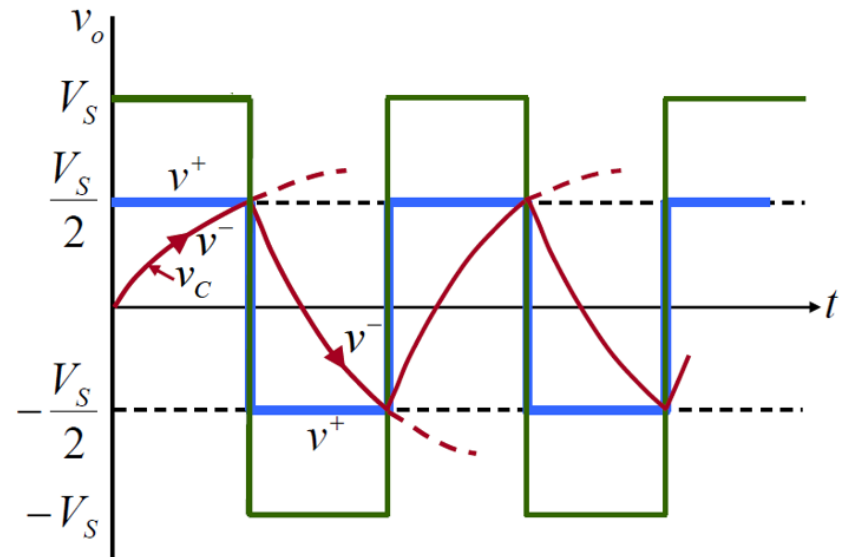
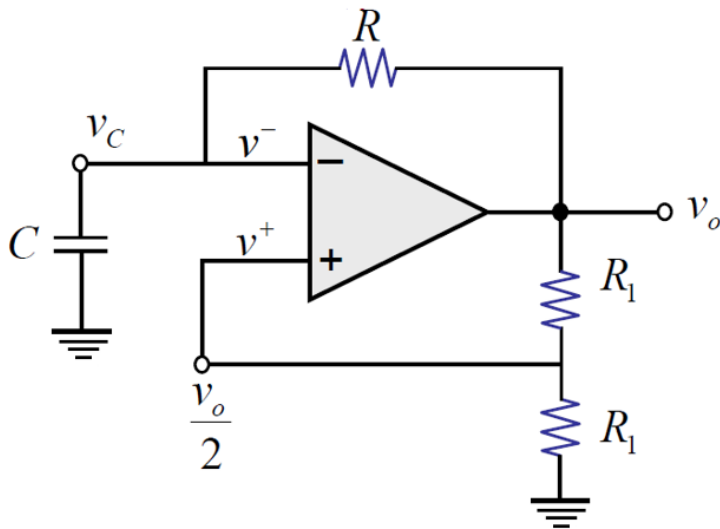


Invert ON

Positive Feedback in Op-Amp Circuits

RC Relaxation Oscillator

The Schmitt trigger circuit (covered at the end of Chapter 6) utilized positive feedback to switch (with hysteresis) between two states (voltage levels). Adding a resistor and a capacitor to this circuit as shown in the figure below results in a simple but elegant oscillator. The oscillator generates a square wave with peak-to-peak voltage equals to V_S . It can be shown that the Hz frequency of oscillation is $f = \frac{1}{2\ln(3)RC}$. ← **Your turn: Derive it.**



Solution of Differential Equations Employing Operational Amplifier Circuits

Consider the first-order ordinary linear differential equation,

$$\dot{y}(t) + 8y(t) = f(t) \quad \text{or} \quad \dot{y}(t) = f(t) - 8y(t)$$

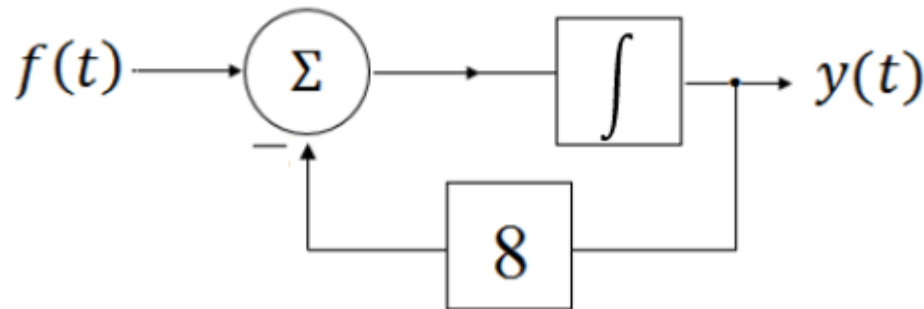
Initial condition $y(0)$

Note: $\dot{y} \equiv \frac{dy}{dt}$

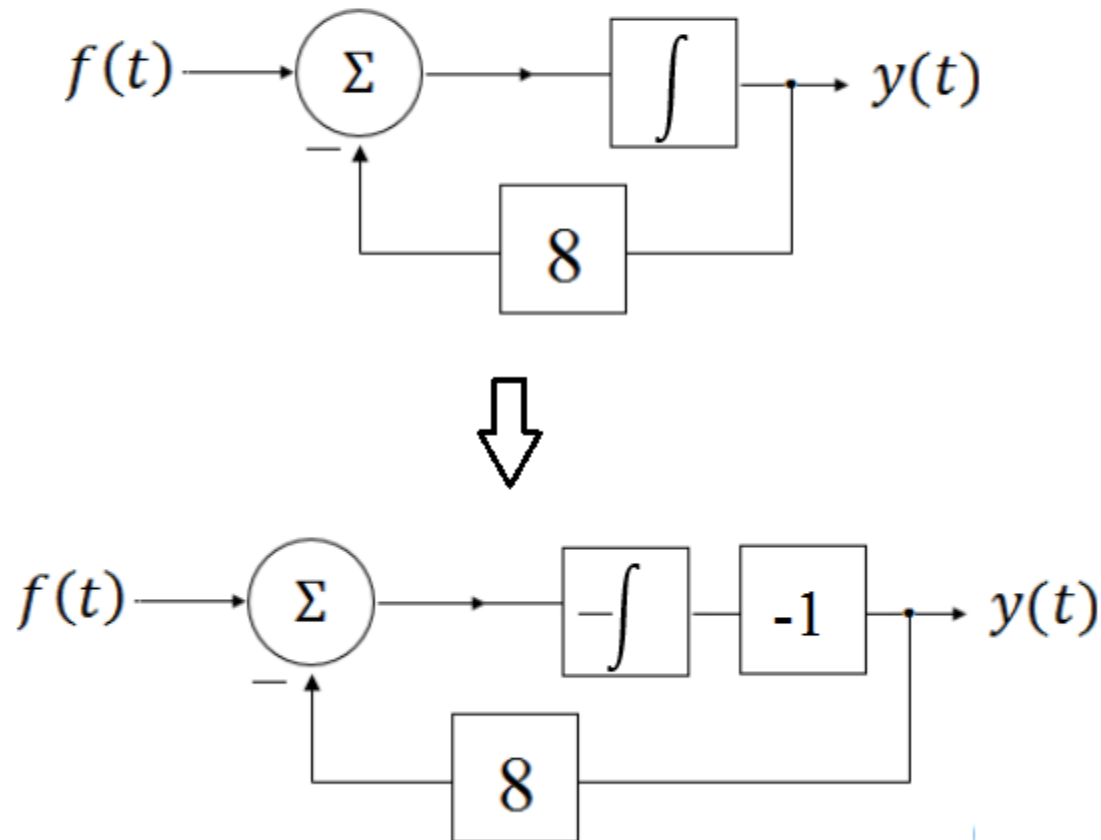
Integrating the above equation leads to,

$$y(t) = \int_0^t (f(\tau) - 8y(\tau)) d\tau$$

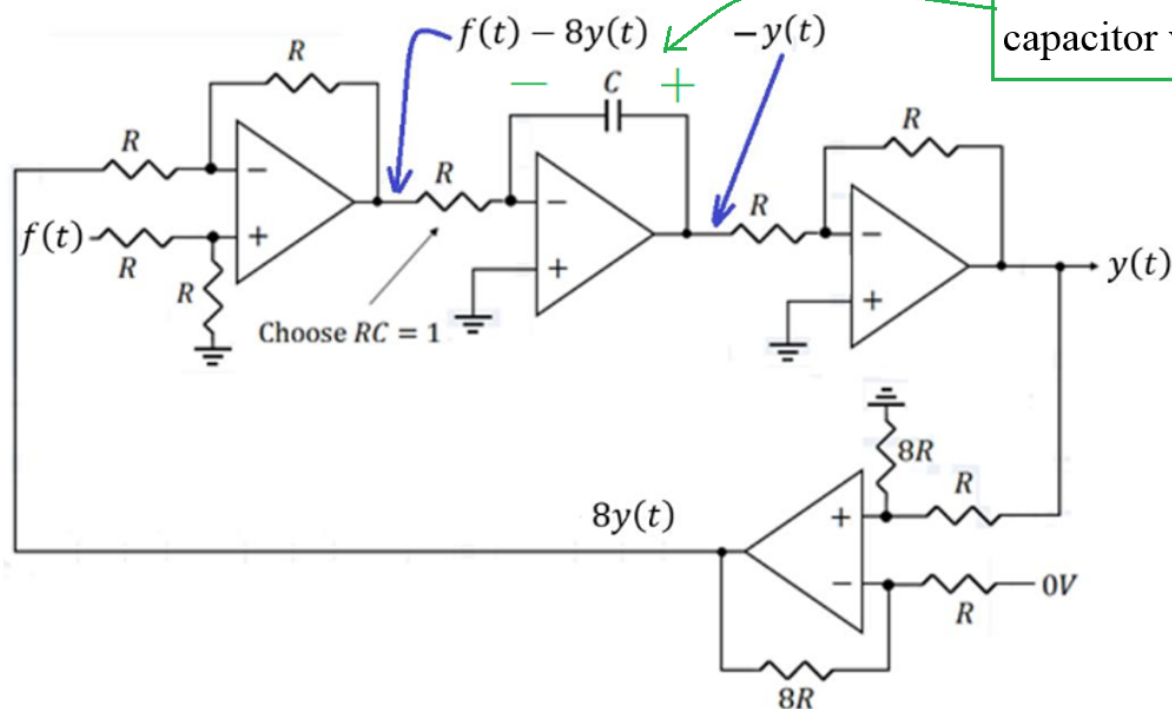
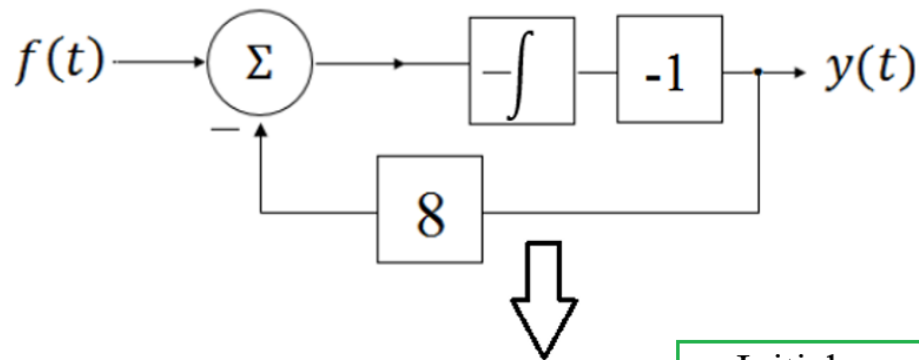
which can be realized as follows [this feedback formulation technique is due to Lord Kelvin (1875)],



Solution of Differential Equations Employing Operational Amplifier Circuits



Solution of Differential Equations Employing Operational Amplifier Circuits

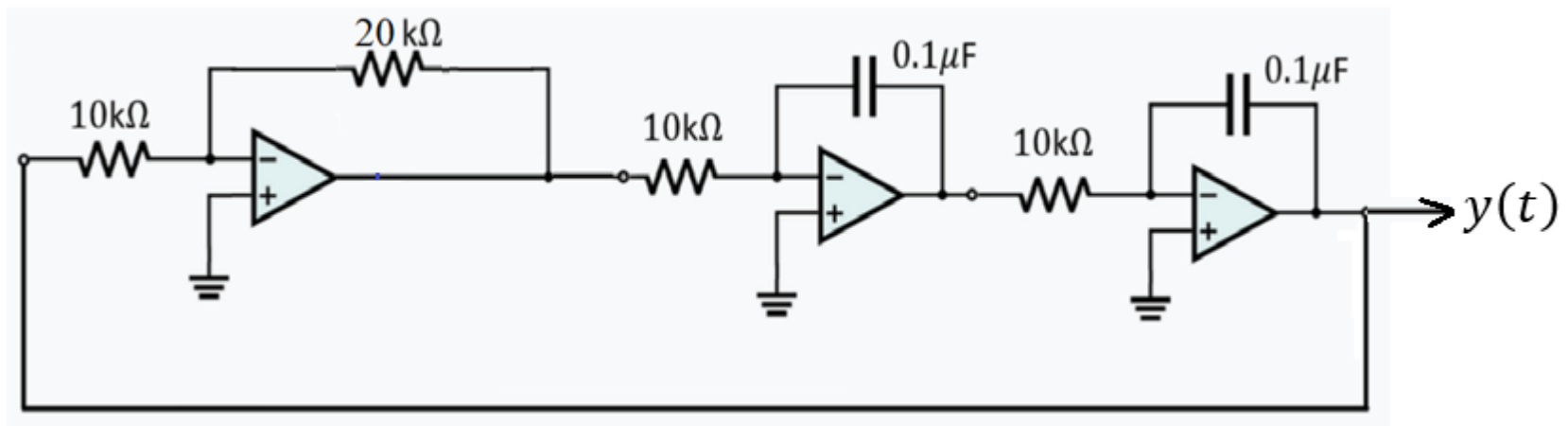


Initial condition is set via the capacitor voltage, $y(0) = -v_C(0)$

Study the example in Section 7.9 (pp. 292-297)

Solution of Differential Equations Employing Operational Amplifier Circuits

Your turn: Determine the differential equation solved by the following circuit.



Circuit Realization of the Lorenz System

Objective: Build an analog circuit consisting of amplifiers, integrators and multipliers that implements the diffusion-less Lorenz chaotic dynamical system, and experimentally generate its phase-plane dynamics.

$$\dot{x} = y - x$$

$$\dot{y} = -xz$$

$$\dot{z} = xy - a$$

This is an example of a nonlinear system that can't be solved analytically. It can be solved by simulation (digital numerical or analog)

[Details](#)
[Multisim Circuit](#)