

ECE 3300
Electrical Circuits: Laboratory
Experiment 10 Pre-Reading
Transient Response of Second-Order Circuits
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In this experiment, the student will examine the properties of inductors and the transient behavior of simple second order circuits i.e., (circuits containing two independent energy storing components).

Learning Objectives

By the end of this unit, the student should be able to:

- Use experimental setup to generate and display on the scope the transient response of a series *RLC* circuit.
- Identify the nature of second-order circuit current/voltage waveforms as: Over-damped, critically-damped, under-damped and oscillatory.
- Build a series *RLC* circuit and adjust its resistance to exhibit its various damping modes.
- Measure the damping factor α of an under-damped series *RLC* circuit.
- Measure the *damped natural frequency* ω_d of an under-damped series *RLC* circuit.

The LC Circuit

A second order circuit consists of two *independent* energy storing elements (e.g., capacitors and/or inductors). “Independent” means that if two capacitors (or two inductors) are present in the circuit, then they are located such that they are not in series or in parallel with each other. The most basic second order circuit is the one that consists of one inductor and one capacitor that are connected in parallel (or series), as shown in Figure 1b.

Consider a charged capacitor of capacitance C and initial voltage V_o and an inductor of inductance L , as shown in Figure 1a. When the switch is open, the capacitor retains an initial energy $E_C(0) = \frac{1}{2}CV_o^2$. On the other hand, the inductor has zero current ($I_o = 0$) and, therefore, its initial energy is $E_L(0) = \frac{1}{2}LI_o^2 = 0$.

When the switch is closed, we obtain the circuit in Figure 1b. Because of capacitor voltage and inductor current continuity constraints, the current in the circuit is zero and the voltage across the capacitor (and across the inductor) is V_o at the instant just after closing the switch. The capacitor's voltage will drive a current in the circuit for $t > 0$. This current flows through the inductor and causes its voltage to change in a way that it remains equal to that of the capacitor. Since the circuit has no resistors (and assuming ideal capacitor and ideal inductor), the circuit will not lose any energy as it operates. This would require that the total energy in the circuit at any time be equal to the initial energy $\frac{1}{2}CV_o^2$. Therefore, we can write $E_C(t) + E_L(t) = \frac{1}{2}CV_o^2$, or

$$\frac{1}{2}Cv_C^2(t) + \frac{1}{2}Li_L^2(t) = \frac{1}{2}CV_o^2, t \geq 0 \quad (1)$$

So, as the capacitor voltage decreases its energy drops. This energy is transferred to the inductor but the total energy in the circuit remains constant.

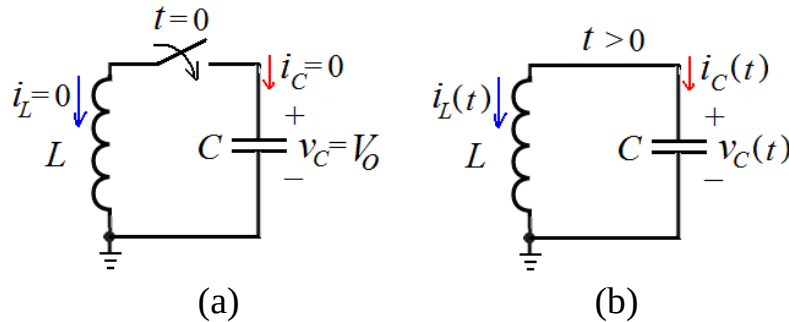


Figure 1. (a) A capacitor with initial voltage V_o ready to be connected to an inductor. (b) The circuit after the switch has been closed.

Another fact is that the energy in the capacitor (and the inductor) can't become constant; i.e., the energy in each component must always be changing. Constant energy implies that the capacitor voltage and the inductor current are constants. This means that for all time the inductor voltage (given by Ldi/dt) is zero and the capacitor voltage (capacitor is in parallel with the inductor) is zero. But, if the capacitor voltage is zero for all time, then its current (Cdv/dt) is zero for all time. This can only happen if the inductor and capacitor current and voltage are zero at the same time, which implies that no energy can exist in the circuit. This contradicts the above assumption of initial energy (initial non-zero voltage) in the capacitor.

It turns out that the initial capacitor energy is gradually transferred to the inductor until the inductor has all that energy stored in it. As soon as that happens, the inductor starts to transfer its energy back to the capacitor until the capacitor recovers all its initial energy. This cycle is then repeated indefinitely.

Let us now solve for the capacitor voltage. Using the relations $i_C(t) = Cdv_C(t)/dt$ and $v_L(t) = Ldi_L(t)/dt$ and noting that for the LC circuit we have $i_C(t) = -i_L(t)$ and $v_C(t) = v_L(t)$, we may write

$$v_C(t) = v_L(t) = \frac{Ldi_L(t)}{dt} = -\frac{Ldi_C(t)}{dt} = -LC \frac{d^2v_C(t)}{dt^2}$$

or

$$\frac{d^2v_C(t)}{dt^2} = -\frac{1}{LC}v_C(t) \quad (2)$$

Equation (2) is a second-order linear differential equation which can be solved for $v_C(t)$ subject to the initial conditions $v_C(0) = V_o$ and $dv_C(0)/dt = i_C(0)/C = -i_L(0)/C = 0$.

Alternatively, one might find the solution by inspection. Equation (2) requires that the solution for $v_C(t)$ has a functional form such that the second derivative of $v_C(t)$ is proportional to $v_C(t)$ and that $v_C(t)$ does not become constant for all time. We know that a sinusoid has this property. Therefore, we try the solution $v_C(t) = A\cos(\omega_o t)$, where ω_o is the radian frequency. Substituting this trial solution in Equation (2) gives

$$-A\omega_o^2 \cos(\omega_o t) = -\frac{A}{LC} \cos(\omega_o t)$$

From which we obtain (by coefficient matching):

$$\omega_o = \frac{1}{\sqrt{LC}} \quad (3)$$

The amplitude A can be solved for by employing the initial condition $v_C(0) = V_o$ in $v_C(t) = A\cos(\omega_0 t)$, which leads to $A = V_o$. Therefore, the capacitor voltage in the LC circuit is

$$v_C(t) = V_o \cos\left(\frac{1}{\sqrt{LC}}t\right) \quad (4)$$

The inductor current can now be determined as $i_L(t) = -i_C(t) = -Cdv_C(t)/dt$, which upon using Equation (4) gives

$$i_L(t) = \frac{V_o}{\sqrt{L/C}} \sin\left(\frac{1}{\sqrt{LC}}t\right) \quad (5)$$

Equations (4) and (5) can now be used to derive the expressions for the capacitor and inductor energy $E_C(t) = \frac{1}{2}C[v_C(t)]^2$ and $E_L(t) = \frac{1}{2}L[i_L(t)]^2$, respectively. Figure 2 shows a plot of the instantaneous energy in the capacitor and the inductor as well as the (constant) total energy predicted by Equation (1).

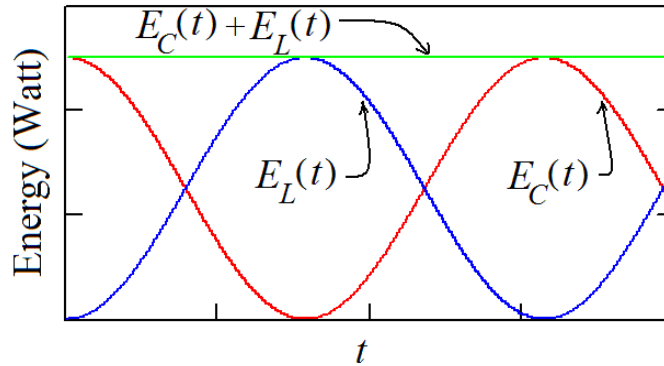


Figure 2. Capacitor and inductor instantaneous energy plots. The total energy at any time is equal to the initial energy in the circuit

An interesting mechanical model for the LC circuit is the simple suspended pendulum shown in Figure 3. The pendulum angle θ (the angle between the arm and the vertical line) is governed by the second-order differential equation (assuming no friction and small θ):

$$\frac{d^2\theta(t)}{dt^2} = -\frac{g}{l}\theta(t) \quad (6)$$

where g is gravitational acceleration and l is the length of the pendulum arm. By comparing Equation (6) to Equation (2), we observe that θ corresponds to $v_C(t)$ and the ratio g/l corresponds to the ratio $1/LC$. Also note that the angular velocity $d\theta/dt$ is proportional to $i_C(t)$ since the capacitor current $i_C(t) = Cdv_C(t)/dt$.

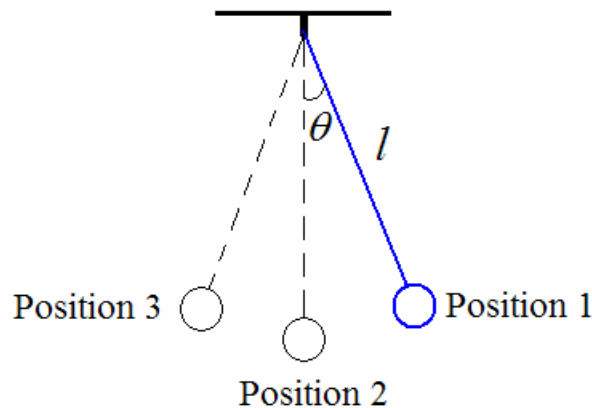


Figure 3. A simple pendulum.

Assuming that the pendulum starts with a (small) initial angle θ and zero angular velocity (Position 1 in Figure 3), then the system initial energy is in the form of potential energy. As the pendulum starts to swing, the potential energy starts to decrease and the kinetic energy begins to increase. When the pendulum is in the vertical position (Position 2), the potential energy becomes zero and all of the system's energy is now in the form of kinetic energy (here, the pendulum has maximum velocity). After that, the kinetic energy begins to decrease and the potential energy of the pendulum begins to increase. All of the energy will ultimately get converted to potential energy as the pendulum reaches its maximum swing position (Position 3). This energy transformation cycle is repeated as the pendulum swings back to its initial position, and so on. With no friction, the pendulum will oscillate for ever.

The Series *RLC* Circuit: Under-Damped Response

The *LC* circuit considered in the previous section is ideal and its oscillatory behavior is true in theory. Any *LC* circuit you build in the lab is actually an *RLC* circuit, because the inductor will always have a small series resistance.

The circuit in Figure 4 takes into account the inductor resistance. Here, R represents the inductor's resistance and any additional resistance that may exist in series with the capacitor and inductor.

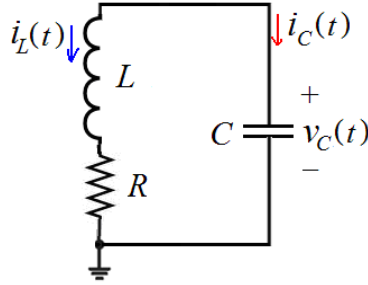


Figure 4. A series RLC circuit.

The analysis of the series RLC circuit can be found in most introductory linear circuits textbooks. Assuming a source-free series RLC circuit, the equation that governs the capacitor voltage is the second order differential equation:

$$\frac{d^2 v_C(t)}{dt^2} + \frac{R}{L} \frac{dv_C(t)}{dt} + \frac{1}{LC} v_C(t) = 0 \quad (7)$$

The solution to this differential equation, assuming $v_C(0) = V_o$ and $i_L(0) = 0$, is given by (the discharge capacitor voltage response):

$$v_C(t) = V_o \sqrt{1 + \left(\frac{\alpha}{\omega_d}\right)^2} e^{-\alpha t} \cos\left[\omega_d t - \tan^{-1}\left(\frac{\alpha}{\omega_d}\right)\right] \quad (8)$$

where the *damping factor* α (also known as the *Neper frequency*, measured in rad/sec) and the *damped radian frequency* ω_d are defined as follows:

$$\alpha = \frac{R}{2L} \quad (9)$$

$$\omega_d = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2} = \sqrt{\omega_o^2 - \alpha^2} \quad (10)$$

For ω_d to be real, we must have $\alpha < \omega_o$; this requires that $R/2L$ be less than $1/\sqrt{LC}$ or $R < 2\sqrt{L/C}$.

A typical capacitor *discharge voltage* and *discharge current* (recall that $i_C = Cdv_C/dt$) for a series *RLC* circuit that meets the above conditions are plotted in Figure 5. More specifically, the circuit has the following parameters: $R = 62\Omega$, $L = 1\text{mH}$, $C = 0.1\mu\text{F}$ and $V_o = 8\text{V}$ (note that $R = 62\Omega$ meets the requirement: $R < 2\sqrt{L/C} = 200$).

The plots in Figure 5 are damped sinusoids. When oscillations show up in the decaying response, then the response is referred to as an *under-damped response* (we also refer to the *RLC* circuit that generates such responses as *under-damped RLC circuit*).

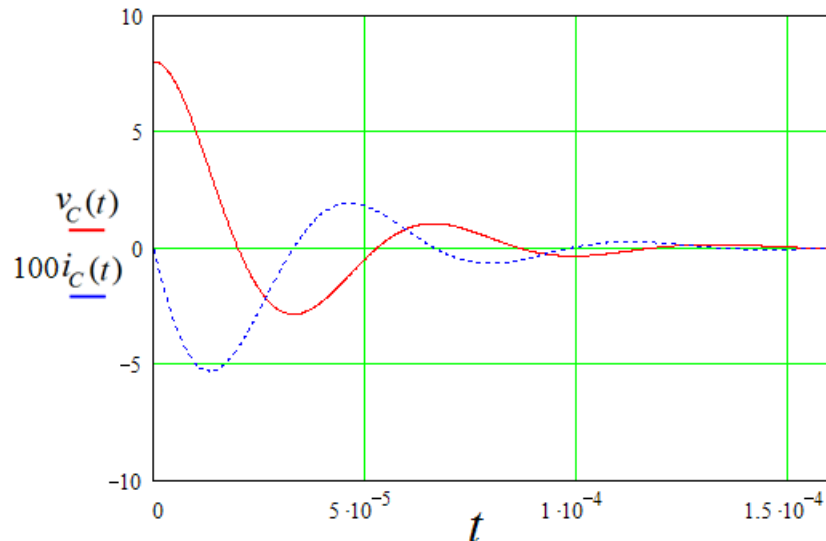
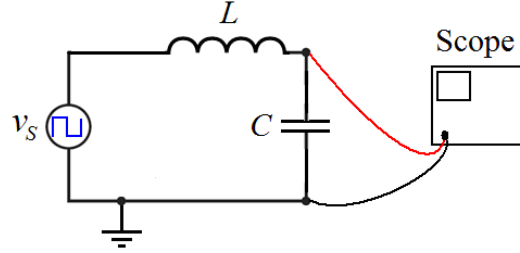


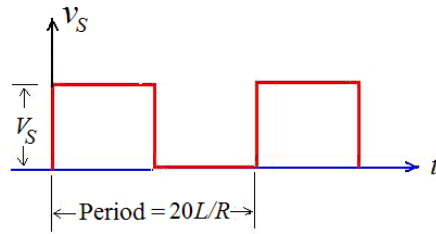
Figure 5. Capacitor voltage and current responses for a source-free, *under-damped RLC* circuit with $R = 62\Omega$, $L = 1\text{mH}$, $C = 0.1\mu\text{F}$ and $v_C(0) = 8\text{V}$.

In the lab, you will be asked to build an *LC* circuit (refer to Figure 6a) and measure its capacitor voltage discharge cycle. You will use the function generator to generate a positive square wave with peak-to-peak value of $V_S = V_o$ Volts (refer to Figure 6b). Actually, your circuit is a series *RLC* circuit where $R = R_o + R_S$; R_o is the inductor's resistance and R_S is the function generator's internal resistance (typically $R_S = 50\Omega$).

If the function generator maintains a voltage V_o for a long enough period, the capacitor will become fully charged (i.e., for large t , $v_C(t)$ approaches V_o); recall that an RLC circuit driven by a DC source will ultimately enter the DC steady-state operation mode whereby the inductor behaves as a short circuit and the capacitor behaves as an open circuit.



(a)



(b)

Figure 6. (a) LC circuit. (b) Function generator signal.

In fact, it can be shown that the capacitor voltage during the *charging cycle*, where $v_s(t) = V_s = V_o$, $v_C(0) = 0$ and $R < 2\sqrt{L/C}$, is given by:

$$v_C(t) = V_o \left\{ 1 - \sqrt{1 + \left(\frac{\alpha}{\omega_d} \right)^2} e^{-\alpha t} \cos \left[\omega_d t - \tan^{-1} \left(\frac{\alpha}{\omega_d} \right) \right] \right\} \quad (11)$$

When t approaches $5/\alpha$, the exponential term $e^{-\alpha t}$ approaches zero. Therefore, according to Equation (11), $v_C(5/\alpha)$ is very close to V_o . This suggests that we set the period, T , of the square wave to $10/\alpha$ or $T = 20L/R$ (i.e., $f = 1/T = R/20L$).

The Multisim solution for the capacitor voltage for an under-damped RLC circuit driven by a positive square wave, with frequency $f = R/20L$, is depicted in Figure 7.

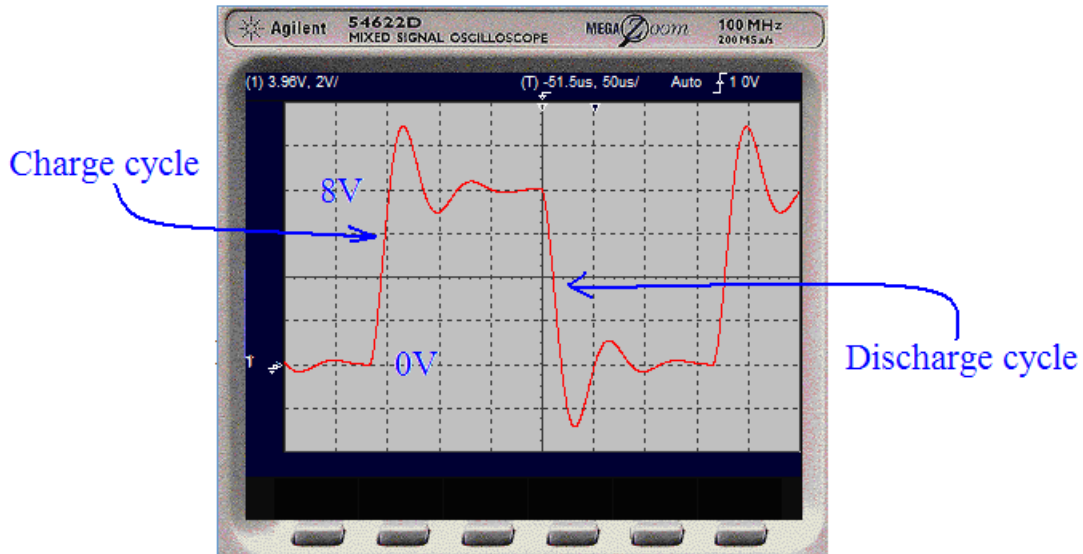


Figure 7. Multisim solution for an under-damped RLC circuit driven by a positive square wave.

The capacitor discharge voltage waveform can be captured and displayed on a scope, as shown in Figure 8 (red plot). The outer envelope of the discharge signal is determined by the decay term $V_o e^{-\alpha t}$ (blue plot). Let us use T to refer to the period of the sinusoid term. Experimentally, T can be measured as the difference between the time of the second peak and that of the first peak of the discharge signal. Using the measured T value, we can determine the damped frequency as $\omega_d = 2\pi/T$.

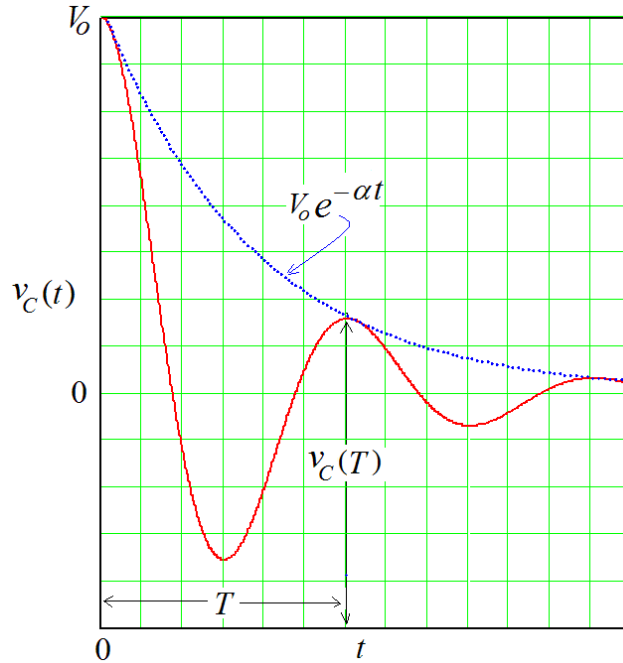


Figure 8. Capacitor discharge waveform in an under-damped series RLC circuit.

The damping factor, α , can also be measured experimentally. Here, the capacitor voltage at $t = T$ is given by the value of the second peak of the discharge voltage, $v_C(T)$ (refer to Figure 8). Now, equating $v_C(T)$ to $V_o e^{-\alpha T}$ and solving for α gives:

$$\alpha = \frac{1}{T} \ln \left(\frac{V_o}{v_C(T)} \right) \quad (12)$$

Therefore, α can be estimated by measuring T and $v_C(T)$ (using the scope's cursor system) and then substituting these measured values in Equation (12).

The Series RLC Circuit: Over-Damped Response

In the preceding section, it was assumed that the series resistance is less than $2\sqrt{L/C}$. In this section, we consider the case where the *total series resistance* is greater than $2\sqrt{L/C}$ which leads to *over-damped response* for the series RLC circuit.

A series RLC circuit driven by a function generator is shown in Figure 9. The total resistance, R_T , is the sum of the resistance R , the inductor resistance R_o and the function generator's resistance R_S :

$$R_T = R + R_o + R_S \quad (13)$$

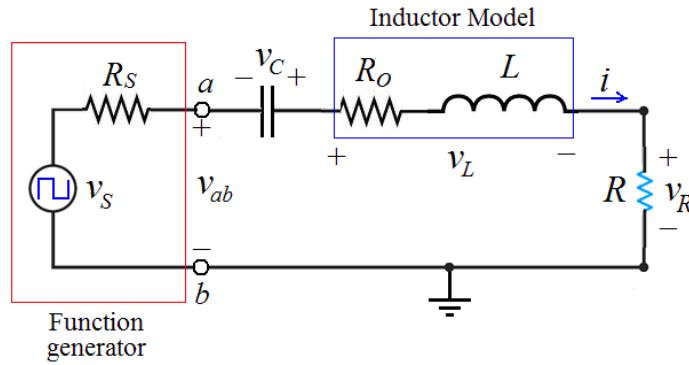


Figure 9. Circuit diagram for an RLC circuit that takes into account: The inductor's resistance and the function generator's internal resistance.

When $v_s(t)$ is a *negative* square wave with amplitude $V_S = -V_o$ and frequency $f = R_T/20L$ (as shown in Figure 10), the circuit in Figure 9 can be reduced to two separate circuits as shown in Figure 11. In the charging circuit in Figure 11a, the square wave applies a voltage $-V_o$ long enough to completely charge the capacitor to $+V_o$ (here, the circuit current will be zero). The circuit in Figure 11b discharges the capacitor's energy. It starts with $i(0) = 0$ and $v_C(0) = V_o$.

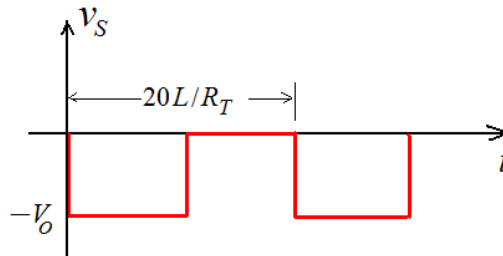


Figure 10. Negative square wave with amplitude $-V_o$ and $f = R_T/20L$.

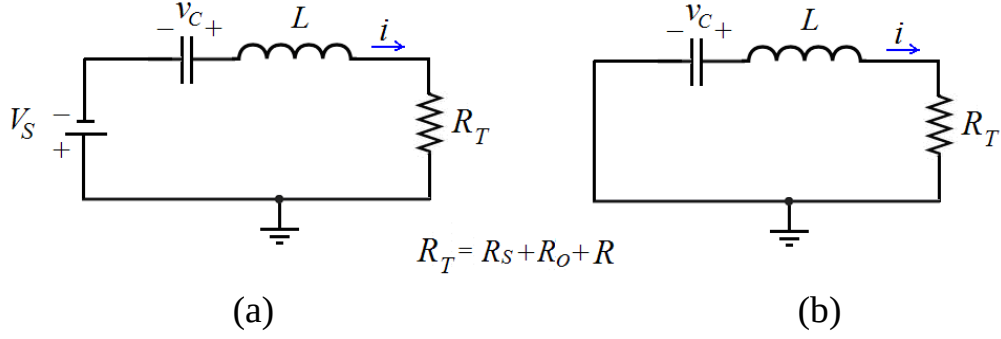


Figure 11. (a) Charging RLC circuit with zero initial energy. (b) Discharging RLC circuit with $i(0) = 0$ and $v_C(0) = V_o$.

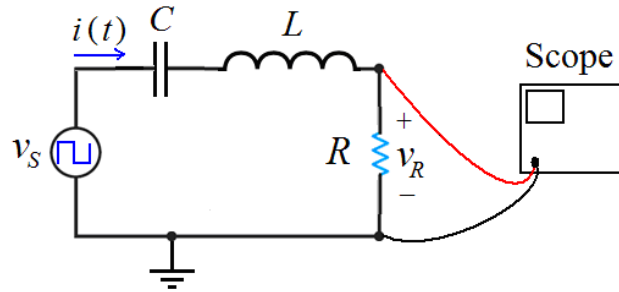


Figure 12. Experimental setup for measuring the series RLC current.

In the lab, a scope can be used to measure the voltage across the resistor, $v_R(t)$, as shown in Figure 12. The circuit current, $i(t)$, can be obtained as $i(t) = v_R(t)/R$.

Theoretically, the circuit current (during the *discharge period* of the RLC circuit with $v_S(t) = 0$) can be obtained by solving the following second-order linear differential equation:

$$\frac{d^2 i(t)}{dt^2} + \frac{R_T}{L} \frac{di(t)}{dt} + \frac{1}{LC} i(t) = 0 \quad (14)$$

subject to the initial conditions $i(0) = 0$ and $v_C(0) = V_o$, and the assumption that $R_T > 2\sqrt{L/C}$. The solution is given by:

$$i(t) = \frac{V_o}{\sqrt{R_T^2 - 4L/C}} \left[e^{-(\alpha - \sqrt{\alpha^2 - \omega_0^2})t} - e^{-(\alpha + \sqrt{\alpha^2 - \omega_0^2})t} \right] \quad (15)$$

where ω_o is given by Equation (3) and $\alpha = R_T/2L$. By knowing $i(t)$, one can find all circuit voltages (refer to Figure 9) according to the following equations:

$$v_R(t) = Ri(t) \quad (16)$$

$$v_{ab}(t) = -R_S i(t) \quad (17)$$

$$v_L(t) = R_o i(t) + L \frac{di(t)}{dt} \quad (18)$$

$$v_C(t) = V_o - \frac{1}{C} \int_0^t i(\tau) d\tau \quad (19)$$

The two exponential terms in Equation (15) are the natural modes of the circuit. There are two damping rates here: $\alpha_1 = \alpha - \sqrt{\alpha^2 - \omega_o^2}$ and $\alpha_2 = \alpha + \sqrt{\alpha^2 - \omega_o^2}$. Since $\alpha_1 < \alpha_2$, the decaying current signal is proportional to the $e^{-\alpha_1 t}$ term for large t . Therefore, in the experimental setup of Figure 12, the frequency of the input square wave voltage signal need to be set to $f = \alpha_1/10$. This will guarantee full charge/discharge current cycles.

The Series *RLC* Circuit: Critically-Damped Response

Finally, we consider the case of a series *RLC* circuit with $R_T = 2\sqrt{L/C}$. This is the case where the response is at the transition point between under-damped and over-damped operation. It can be shown that the circuit current during the discharge period of this *critically-damped* circuit is given by:

$$i(t) = \frac{V_o}{L} t e^{-\frac{R_T}{2L} t} = \frac{V_o}{L} t e^{-\frac{1}{\sqrt{LC}} t} \quad (21)$$

Equations (16)-(19) also apply for this circuit.

Experimental Setup for Observing Various Types of Damping

In the laboratory, you will build the circuit in Figure 13 (with the supply voltage shown in Figure 14) that utilizes a variable resistor (employing a potentiometer) for R . By rotating a potentiometer's knob you will be able to change the resistance so as to obtain and display the under-damped, critically-damped and over-damped response for the capacitor voltage.

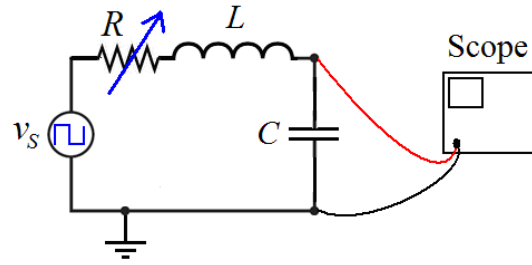


Figure 13. A series RLC circuit employing a potentiometer as a variable resistor.

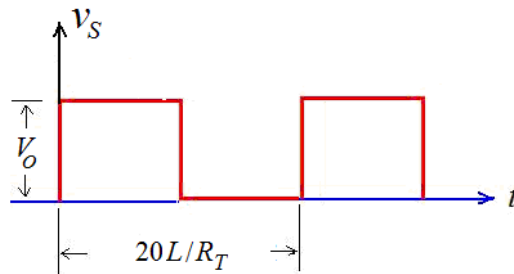


Figure 14. Function generator waveform for the circuit in Figure 13.

Figure 15 depicts theoretical waveforms for the capacitor voltage in a series RLC circuit for various resistance values. Equation (8) was used to obtain the under-damped response ($R_T < 2\sqrt{L/C}$). Equations (15) and (19) were used to obtain the over-damped response ($R_T > 2\sqrt{L/C}$). Equations (19) and (21) were used to obtain the critically-damped response ($R_T = 2\sqrt{L/C}$). Finally, Equation (4) was used to generate the oscillatory response ($R_T = 0$). Note: This response can't be obtained using the experimental setup shown in Figure 13 with $R = 0$ (why?).

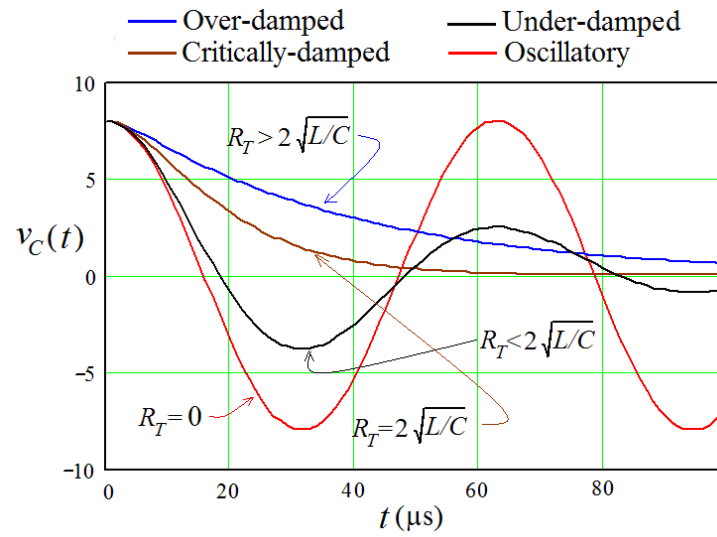


Figure 15. Capacitor voltage waveforms for a series RLC circuit for various resistance values.