

ECE 3310
Electrical Circuits: Laboratory
Experiment 12 Pre-Reading
Electric Filters

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In this experiment, the student will experiment with various types of electric filters.

Learning Objectives

By the end of this unit, the student should be able to experimentally:

- Build and test a low-pass filter.
- Build and test a high-pass filter.
- Build and test a band-pass filter.
- Build and test a notch-pass filter.
- Generate the magnitude and phase of a filter's transfer function.
- Determine the cut-off frequency for an electric filter.

The Transfer Function

Consider a linear circuit operating in the AC steady-state mode with sinusoidal input voltage $v_{in}(t)$ and sinusoidal output voltage $v_{out}(t)$ oscillating at the same frequency. The output voltage is typically the voltage across a resistor, an inductor or a capacitor. A conceptual phasor domain equivalent circuit is shown in Figure 1. Here, $\mathbf{V}_{in}$ and $\mathbf{V}_{out}$ represent the phasor voltage equivalents of v_{in} and v_{out} , respectively.

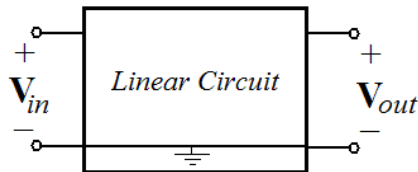


Figure 1. A conceptual phasor-domain linear circuit with input voltage $\mathbf{V}_{in}$ and output voltage $\mathbf{V}_{out}$.

Let us define the *complex valued* function $\mathbf{H}(\omega_o)$ as the ratio of the phasors $\mathbf{V}_{out}$ and $\mathbf{V}_{in}$:

$$\mathbf{H}(\omega_o) = \frac{\mathbf{V}_{out}}{\mathbf{V}_{in}} \quad (1)$$

Where ω_o is the frequency of the sinusoid $v_{in}(t)$. The *transfer function* $\mathbf{H}(\omega_o)$ can be obtained by employing standard circuit analysis methods. Employing Equation (1), one can obtain the output phasor voltage as:

$$\mathbf{V}_o = \mathbf{H}(\omega_o)\mathbf{V}_{in} \quad (2)$$

Sinusoidal Expansion of a Periodic Signal

According to a powerful mathematical theorem (the Fourier Series Expansion Theorem), any *physical* periodic signal $v(t)$ with frequency ω_o (say, a square wave, a triangular wave, etc.) can be written as the sum of a constant term and an infinite number of sinusoid terms; each sinusoid has a unique frequency that is an integer multiple of ω_o . Formally, we write:

$$v(t) = V_0 + V_1 \cos(\omega_o t + \theta_1) + V_2 \cos(2\omega_o t + \theta_2) + V_3 \cos(3\omega_o t + \theta_3) + \dots \quad (3)$$

The cosine terms in the *Fourier series expansion* of Equation (3) are called “harmonics.” The constant term V_0 represents the signal average value or DC value. In general, the amplitudes of higher harmonic terms are relatively small. Therefore, the periodic signal $v(t)$ can be well approximated by its DC term and the first several harmonic terms.

As an example, consider a positive square wave voltage signal $v_{in}(t)$ with amplitude A and frequency ω_o . This signal can be expressed as the Fourier series:

$$v_{in}(t) = \frac{A}{2} + \frac{2A}{\pi} \cos(\omega_o t) - \frac{2A}{3\pi} \cos(3\omega_o t) + \frac{2A}{5\pi} \cos(5\omega_o t) - \frac{2A}{7\pi} \cos(7\omega_o t) + \frac{2A}{9\pi} \cos(9\omega_o t) \dots \quad (4)$$

Note how the amplitudes of the higher harmonics converge toward zero. A plot of $v_{in}(t)$ and its first 3-term and 5-term approximation (including the DC term) is shown in Figure 2. The more terms we add the more accurate is the approximation; the higher harmonics improve the approximation of the signal at its sharp transition points. So, we can say that *the sharp transitions in a signal (e.g., the corners of the square wave) are associated with the higher frequency components of (the Fourier series) of a given signal.*

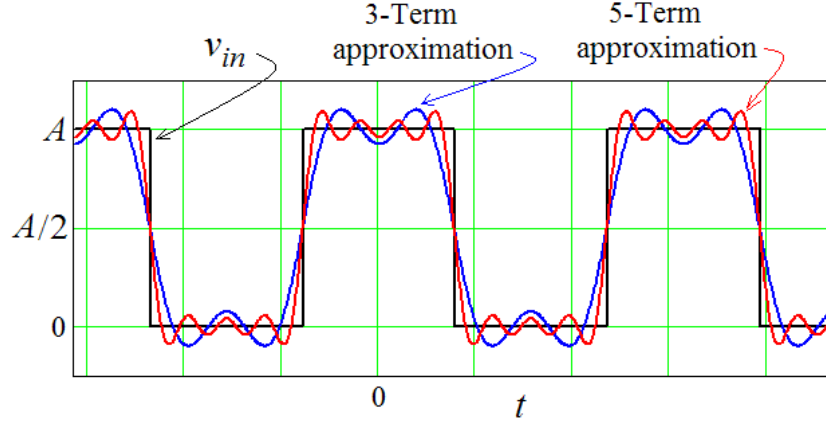


Figure 2. A square wave signal and its 3-term and 5-term Fourier series approximation.

Electric Filters

The transfer function, $\mathbf{H}(\omega_o)$, is a property of the circuit and the frequency, ω_o , of the input; but, it is not dependent on the “shape” of the signal $v_{in}(t)$. Equation (2) holds for the more general case where $v_{in}(t)$ is a periodic, non-sinusoidal signal. In this case, $v_{in}(t)$ can be approximated by the sum of the first several terms of its Fourier series expansion [refer to Equation (3)]:

$$v_{in}(t) = V_0 + V_1 \cos(\omega_o t + \theta_1) + V_2 \cos(2\omega_o t + \theta_2) + V_3 \cos(3\omega_o t + \theta_3) + \dots \quad (5)$$

The steady-state response, $v_{out}(t)$, of a linear circuit to the periodic input in Equation (5) can be obtained by applying the superposition principle: $v_{out}(t)$ is the sum of the responses of the circuit to the various terms in Equation (5), each applied separately. Equivalently, the phasor output voltage, $\mathbf{V}_{out}$, is obtained, using Equation (2), as:

$$\mathbf{V}_{out} = \mathbf{H}(0)V_0 + \mathbf{H}(\omega_o)V_1 \angle \theta_1 + \mathbf{H}(2\omega_o)V_2 \angle \theta_2 + \mathbf{H}(3\omega_o)V_3 \angle \theta_3 + \dots \quad (6)$$

Recalling that $\mathbf{H}(\omega)$ is a complex quantity, we express $\mathbf{H}(\omega)$ as a phasor, $\mathbf{H}(\omega) = |\mathbf{H}(\omega)| \angle \mathbf{H}(\omega)$, and use it to rewrite Equation (6) as:

$$\mathbf{V}_{out} = \mathbf{H}(0)V_0 + |\mathbf{H}(\omega_o)| \angle \mathbf{H}(\omega_o) V_1 \angle \theta_1 + |\mathbf{H}(2\omega_o)| \angle \mathbf{H}(2\omega_o) V_2 \angle \theta_2 + |\mathbf{H}(3\omega_o)| \angle \mathbf{H}(3\omega_o) V_3 \angle \theta_3 \dots \quad (7)$$

The time-domain output voltage, $v_{out}(t)$, can now be written as:

$$v_{out}(t) = \mathbf{H}(0)V_0 + |\mathbf{H}(\omega_o)| V_1 \cos(\omega_o t + \angle \mathbf{H}(\omega_o) + \theta_1) + |\mathbf{H}(2\omega_o)| V_2 \cos(2\omega_o t + \angle \mathbf{H}(2\omega_o) + \theta_2) + |\mathbf{H}(3\omega_o)| V_3 \cos(3\omega_o t + \angle \mathbf{H}(3\omega_o) + \theta_3) \dots \quad (8)$$

Equation (8) has a very interesting and useful interpretation. The circuit acts as a *filter* that can pass through certain *harmonics* of the input signal and attenuate other harmonics. The filter characteristics of several electric circuits are explored in the following sections.

Low-Pass Filter

Consider a series RC circuit with AC input and assume that the circuit is operating in the steady-state mode. The RC phasor circuit is depicted in Figure 3. Here, ω is given by the source's radian frequency.

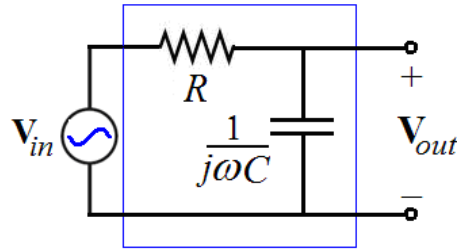


Figure 3. RC low-pass filter.

Before formally solving for the transfer function, it would be instructive to predict the behavior of this circuit for extreme values of ω , namely $\omega = 0$ and $\omega \rightarrow \text{infinity}$. First, for $\omega = 0$, the capacitor's steady-state behavior is that of an open circuit. This leads to zero current in the resistor which, in turn, leads to $\mathbf{V}_{out} = \mathbf{V}_{in}$. On the other hand, as $\omega \rightarrow \text{infinity}$ the capacitor's steady-state behavior is that of a short circuit; this leads to $\mathbf{V}_{out} = 0$. Therefore, the circuit seems to pass inputs with low frequencies and block those with high frequencies. These are the characteristics of a *low-pass filter*.

The output of the circuit is taken to be the voltage across the capacitor, and is given by:

$$\mathbf{V}_{out} = \frac{\mathbf{Z}_C}{R + \mathbf{Z}_C} \mathbf{V}_{in} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} \mathbf{V}_{in} = \left[\frac{1}{\sqrt{1 + (\omega RC)^2}} \angle \left(\tan^{-1} \left(\frac{1}{\omega RC} \right) - \frac{\pi}{2} \right) \right] \mathbf{V}_{in} \quad (9)$$

By Matching Equation (9) to Equation (2), the transfer function for this circuit is given by (in phasor form):

$$\mathbf{H}(\omega) = \frac{1}{\sqrt{1 + (\omega RC)^2}} \angle \left(\tan^{-1} \left(\frac{1}{\omega RC} \right) - \frac{\pi}{2} \right) \quad (10)$$

The magnitude and phase (angle) of $\mathbf{H}$ are given by:

$$|\mathbf{H}(\omega)| = \frac{1}{\sqrt{1 + (\omega RC)^2}} \quad (11)$$

and

$$\angle \mathbf{H}(\omega) = \tan^{-1} \left(\frac{1}{\omega RC} \right) - \frac{\pi}{2} \quad (12)$$

It is instructive to take a look at the plots of $|\mathbf{H}(\omega)|$ and $\angle \mathbf{H}(\omega)$ versus ω . In practice, these plots are generated where ω is displayed on a *log* scale (so as to compress the large range of frequencies). Also, the *decibel* value of $|\mathbf{H}(\omega)|$, written as $|\mathbf{H}(\omega)|_{\text{dB}}$, is used. The $|\mathbf{H}(\omega)|_{\text{dB}}$ value is defined as:

$$|\mathbf{H}(\omega)|_{\text{dB}} = 20 \log(\mathbf{H}(\omega)) \quad (13)$$

The magnitude and phase plots for $\mathbf{H}(\omega)$ are shown in Figure 4 for the values $R = 1\text{k}\Omega$ and $C = 0.1\mu\text{F}$. The magnitude plot shows low attenuation for a low-frequency input and high attenuation for a high frequency input. The filter is then a low-pass filter.

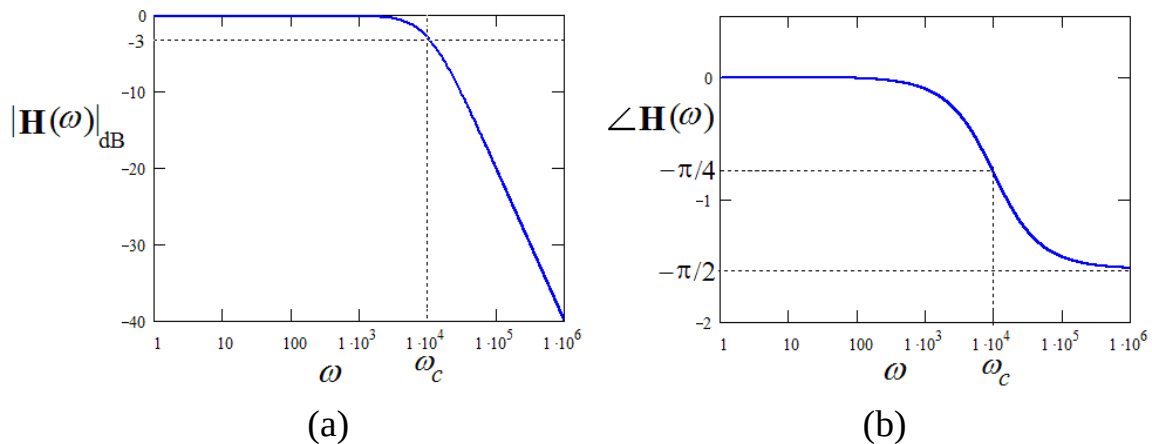


Figure 4. (a) Magnitude (in dB) plot and (b) phase plot for the low-pass filter in Figure 3 (with $R = 1\text{k}\Omega$ and $C = 0.1\mu\text{F}$).

It is useful to define a threshold frequency or “cut-off” frequency for the filter. The *cut-off frequency* is defined as the frequency ω_c at which $|\mathbf{H}(\omega_c)|_{\text{dB}} = -3\text{dB}$. Solving Equation (13) for $|\mathbf{H}(\omega_c)|$, we get:

$$|\mathbf{H}(\omega_c)| = 10^{\frac{|\mathbf{H}(\omega_c)|_{\text{dB}}}{20}} \quad (14)$$

Now, by setting $|\mathbf{H}(\omega_c)|_{\text{dB}} = -3\text{dB}$ in Equation (14) we get $|\mathbf{H}(\omega_c)| = 0.708$. Therefore, *the cut-off frequency is also the frequency at which the amplitude of the filter output is about $1/\sqrt{2}$ of the input signal amplitude*. Using Equation (11), setting $|\mathbf{H}(\omega_c)| = 1/\sqrt{2}$ and solving, we obtain $\omega_c = 1/RC$. The phase $\angle \mathbf{H}(\omega_c)$ can be solved for by substituting $1/RC$ for ω in Equation (12) to obtain $\angle \mathbf{H}(\omega) = -\pi/4$.

Multisim allows the user to generate a filter’s [magnitude](#) and [phase](#) plots by employing the **Bode Plotter Tool**.

The [Multisim solution](#) for the output voltage is shown in Figure 5, for the case where the input signal is a positive square wave with amplitude of 4V and frequency $f = 1\text{kHz}$. It is interesting to note how the filter attenuates the high frequency components associated with the sharp transitions in the input waveform. On the other hand, the filter passes through (with minimal attenuation) the low frequency components associated with the low variation voltage regions of the input waveform.

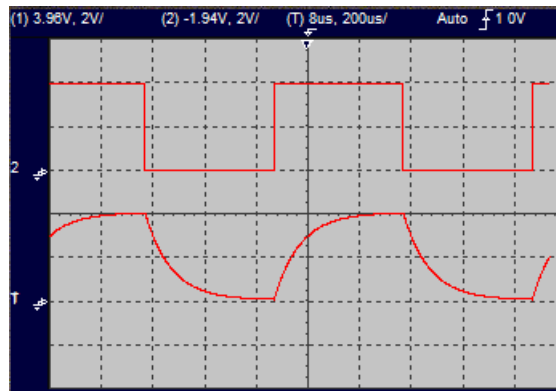


Figure 5. Response of a low-pass filter to a positive square wave input.

The response of the series RC circuit to a positive square wave input (in Figure 5) should be familiar to the student; it was covered in Experiment 8 in the context of the charge/discharge cycle of a capacitor. It is interesting to see how the interpretation of a capacitor *transient* charging/discharging cycle can be viewed from an AC *steady-state* signal filtering point of view. The following example sheds additional light on this issue.

A 1kHz (i.e., $\omega_0 = 2000\pi$ rad/sec) positive square wave with amplitude 4V has the following Fourier series representation:

$$v_{in}(t) = 2 + \frac{8}{\pi} \cos(2000\pi t) - \frac{8}{3\pi} \cos(6000\pi t) + \frac{8}{5\pi} \cos(10000\pi t) - \frac{8}{7\pi} \cos(14000\pi t) + \dots$$

This signal is applied to the low-pass filter of Figure 3 (with $R = 1\text{k}\Omega$ and $C = 0.1\mu\text{F}$). The output of the filter is given by Equation (8). Here, the magnitude and phase of $\mathbf{H}(\omega)$ can be evaluated at the proper harmonic frequency using Equations (11) and (12). Considering only the first four terms in the expansion of $v_{in}(t)$ and employing Equation (8), we obtain the following expression for the output signal:

$$v_{out}(t) \cong 2 + 2.16 \cos(2000\pi t - 0.56) - 1.19 \cos(6000\pi t - 1.08) + 0.77 \cos(10000\pi t - 1.26)$$

The signal $v_{out}(t)$ is plotted in Figure 6. The solution for $v_{out}(t)$, assuming that the first six terms of the $v_{in}(t)$ expansion have been included, is shown in Figure 7.

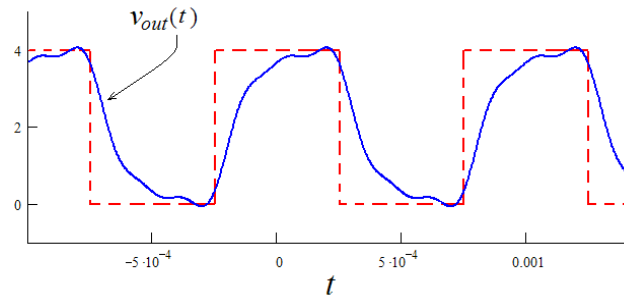


Figure 6. RC low-pass filter response obtained by approximating the square wave input voltage by its first four Fourier series terms. The filter's cut-off frequency is $\omega_c = 1/RC = 10,000$ rad/s and the square wave frequency is $\omega_0 = 2000\pi$ rad/s.

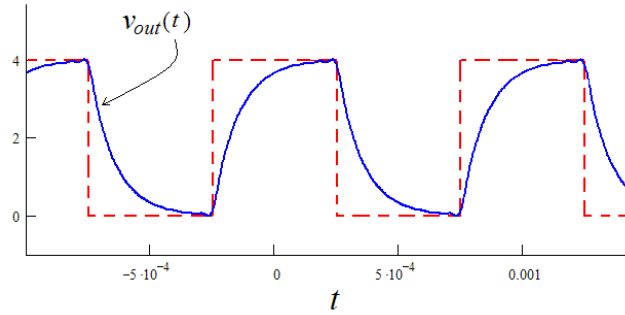


Figure 7. *RC* low-pass filter response obtained by approximating the square wave input voltage by its first six Fourier series terms.

High-Pass Filter

Consider a series *RC* circuit with AC input and assume that the circuit is operating in the steady-state mode. The *RC* phasor circuit is depicted in Figure 8.

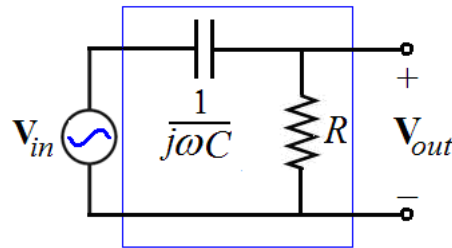


Figure 8. *RC* high-pass filter.

Let us take a qualitative look at the behavior of this circuit for extreme values of source frequency ω , namely $\omega = 0$ and $\omega \rightarrow \text{infinity}$. First, for $\omega = 0$, the capacitor's steady-state behavior is that of an open circuit. This leads to zero current in the resistor which, in turn, leads to $\mathbf{V}_{out} = 0$. On the other hand, as $\omega \rightarrow \text{infinity}$ the capacitor's steady-state behavior is that of a short circuit; this leads to $\mathbf{V}_{out} = \mathbf{V}_{in}$. Therefore, the circuit seems to pass inputs with high frequencies and block those with low frequencies. These are the characteristics of a *high-pass* filter.

The output of the circuit is taken to be the voltage across the resistor, and is given by:

$$\mathbf{V}_{out} = \frac{R}{R + \mathbf{Z}_C} \mathbf{V}_{in} = \frac{R}{R + \frac{1}{j\omega C}} \mathbf{V}_{in} = \left[\frac{1}{\sqrt{1 + \frac{1}{(\omega RC)^2}}} \angle \tan^{-1}\left(\frac{1}{\omega RC}\right) \right] \mathbf{V}_{in} \quad (15)$$

The transfer function for this circuit can be obtained by matching Equations (2) and (15). The result is:

$$\mathbf{H}(\omega) = \frac{1}{\sqrt{1 + \frac{1}{(\omega RC)^2}}} \angle \tan^{-1}\left(\frac{1}{\omega RC}\right) \quad (16)$$

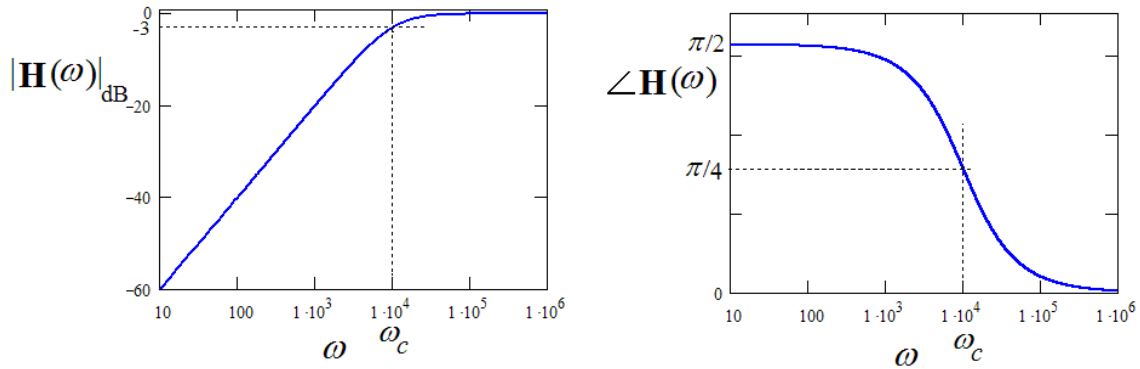
From which the magnitude and phase of $\mathbf{H}(\omega)$ are obtained as:

$$|\mathbf{H}(\omega)| = \frac{1}{\sqrt{1 + \frac{1}{(\omega RC)^2}}} \quad (17)$$

and

$$\angle \mathbf{H}(\omega) = \tan^{-1}\left(\frac{1}{\omega RC}\right) \quad (18)$$

Multisim allows the user to generate the filter's [magnitude](#) and [phase](#) plots by employing the **Bode Plotter Tool**. The magnitude and phase plots for $\mathbf{H}(\omega)$ are shown in Figure 9, for the values $R = 1\text{k}\Omega$ and $C = 0.1\mu\text{F}$. The magnitude plot shows high attenuation for low-frequency inputs and low attenuation for high frequency inputs. The filter is then a *high-pass* filter.



(b)

(b)

Figure 9. (a) Magnitude (dB) plot and (b) phase plot for the high-pass filter in Figure 8 (with $R = 1\text{k}\Omega$ and $C = 0.1\mu\text{F}$).

Using Equation (17), setting $|\mathbf{H}(\omega)| = 1/\sqrt{2}$ and solving for ω , we obtain the cut-off frequency, $\omega_c = 1/RC = 10,000$ rad/sec (about 1.59kHz). The phase $\angle |\mathbf{H}(\omega_c)|$ can be solved for by substituting $1/RC$ for ω in Equation (18) to obtain $\angle \mathbf{H}(\omega) = \pi/4$.

The [Multisim solution](#) for the output voltage is shown in Figure 10, for the case where the input signal is a positive square wave with amplitude of 4V and frequency $f = 1\text{kHz}$. It is interesting to note how the filter attenuates the low frequency components associated with the low-variation voltage regions in the input waveform. On the other hand, the filter passes through the high frequency components associated with the sharp transitions of the input waveform.

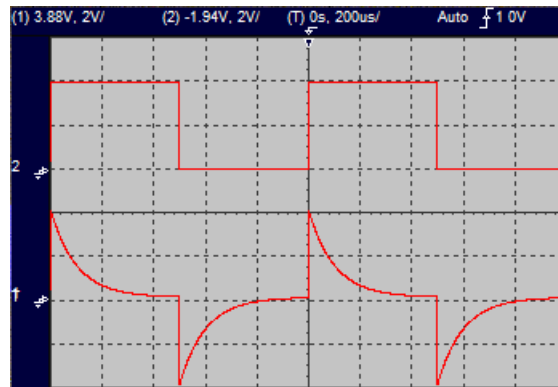


Figure 10. Response of the high-pass filter to a positive square wave input.

Band Pass Filter

Consider the AC steady-state series RLC circuit shown in Figure 11.

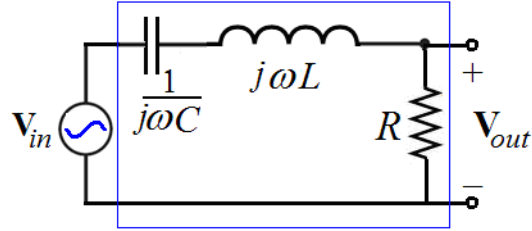


Figure 11. RLC band-pass filter.

Using voltage division, the output voltage can be expressed as:

$$\mathbf{V}_{out} = \frac{R}{R + \frac{1}{j\omega C} + j\omega L} \mathbf{V}_{in} = \left[\frac{R}{\sqrt{R^2 + \left(\frac{\omega^2 LC - 1}{\omega C} \right)^2}} \angle -\tan^{-1} \left(\frac{\omega^2 LC - 1}{\omega RC} \right) \right] \mathbf{V}_{in} \quad (19)$$

The above equation suggests that the transfer function $\mathbf{H}(\omega)$ is:

$$\mathbf{H}(\omega) = \frac{R}{\sqrt{R^2 + \left(\frac{\omega^2 LC - 1}{\omega C} \right)^2}} \angle -\tan^{-1} \left(\frac{\omega^2 LC - 1}{\omega RC} \right) \quad (20)$$

The magnitude and phase of $\mathbf{H}(\omega)$ are given by:

$$|\mathbf{H}(\omega)| = \frac{R}{\sqrt{R^2 + \left(\frac{\omega^2 LC - 1}{\omega C} \right)^2}} \quad (21)$$

and

$$\angle \mathbf{H}(\omega) = -\tan^{-1} \left(\frac{\omega^2 LC - 1}{\omega RC} \right) \quad (22)$$

The [magnitude](#) and [phase](#) plots for $\mathbf{H}(\omega)$ are shown in Figure 12a and 12b, respectively ($R = 1\text{k}\Omega$, $L = 1\text{mH}$ and $C = 0.1\mu\text{F}$ were assumed). It is interesting to note that no attenuation occurs at the resonance frequency $\omega_o = 1/\sqrt{LC}$. Also, at the resonance frequency, $v_{out}(t)$ is in phase with $v_{in}(t)$.

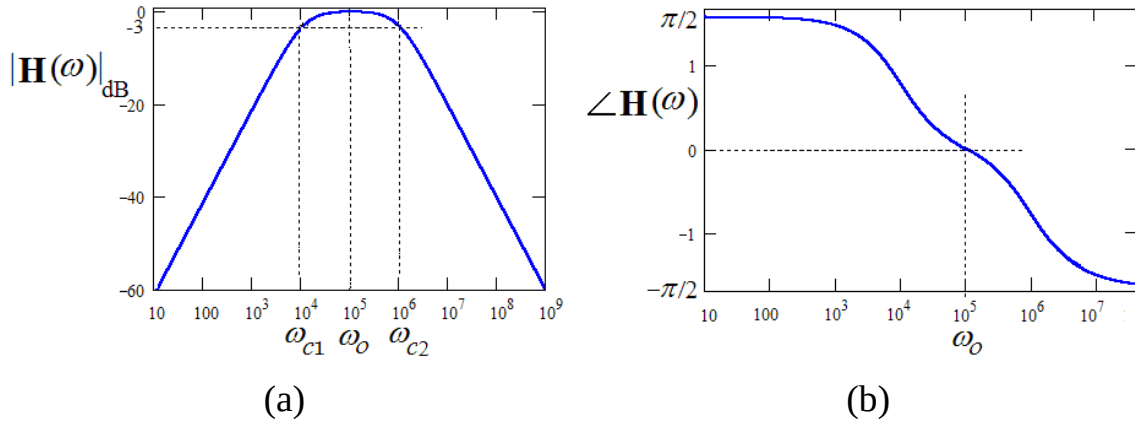


Figure 12. (a) Magnitude and (b) phase plots for the band-pass filter in Figure 11.

The cut-off frequencies ω_{c1} and ω_{c2} are obtained as the solution to the following equation:

$$|\mathbf{H}(\omega)| = \frac{R}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} = \frac{1}{\sqrt{2}} \quad (23)$$

The solution for Equation (23) is:

$$\omega_{c1,c2} = \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} \mp \frac{R}{2L} \quad (24)$$

The band-pass filter is used to pass the frequency components of an input signal that lie between the cut-off frequencies, and attenuate all other frequency components. Figure 13 shows the results of Multisim simulation for a band-pass filter with $R = 500\Omega$, $L = 1\text{mH}$ and $C = 0.1\mu\text{F}$ and a positive square wave input with 4V amplitude and frequency $\omega_o = 1/\sqrt{LC}$.

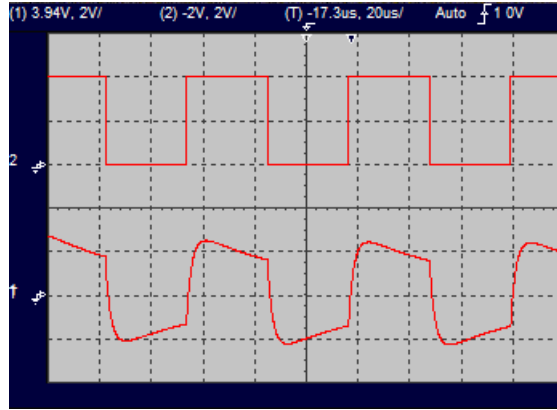


Figure 13. Response of the band-pass filter to a positive square wave input.

Notch Filter

A notch filter is shown in Figure 14. This filter is designed to block a specific frequency component of an input signal.

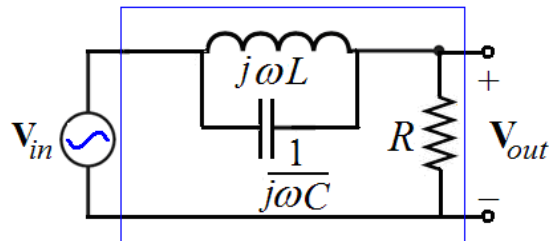


Figure 14. A notch filter

The transfer function for the notch filter in Figure 14 is given by:

$$\mathbf{H}(\omega) = \frac{R}{R + \frac{1}{j\omega C + \frac{1}{j\omega L}}} = \frac{R}{R + j \frac{\omega L}{1 - \omega^2 LC}} \quad (25)$$

The magnitude and phase of $\mathbf{H}(\omega)$ are given by:

$$|\mathbf{H}(\omega)| = \frac{R}{\sqrt{R^2 + \left(\frac{\omega L}{1 - \omega^2 LC} \right)^2}} \quad (26)$$

and

$$\angle \mathbf{H}(\omega) = -\tan^{-1}\left(\frac{\omega L}{R(1 - \omega^2 LC)}\right) \quad (27)$$

The magnitude and phase plots for $\mathbf{H}(\omega)$ are shown in Figure 15a and 14b, respectively ($R = 1\text{k}\Omega$, $L = 1\text{mH}$ and $C = 0.1\mu\text{F}$ were assumed). It is interesting to note that (theoretically) full attenuation occurs at the resonance frequency $\omega_o = 1/\sqrt{LC}$.

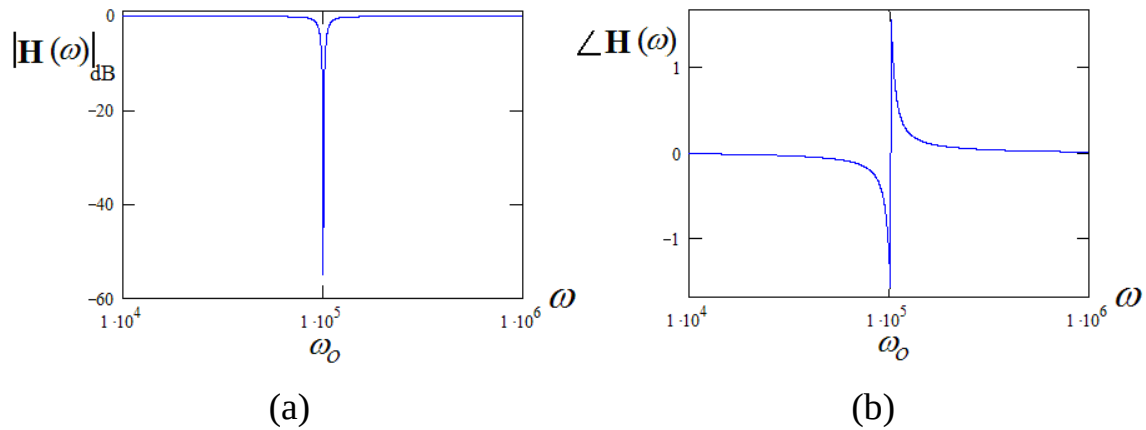


Figure 15. (a) Magnitude and (b) phase plots for the notch filter in Figure 14.

The output of the notch filter for a positive square wave input [4V amplitude and $f = 1/(2\pi\sqrt{LC})$] is shown in Figure 16.

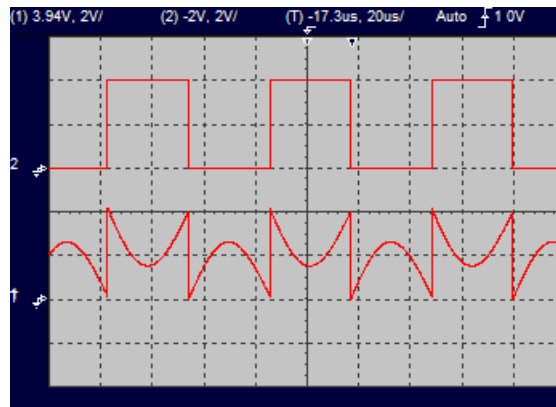


Figure 16. Response of the notch filter to a positive square wave input.