On π, e and $\sqrt{-1}$

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Introduction

The numbers π (approximately equal to 3.14159265) and e (approximately equal to 2.71828183) are irrational (can not be represented as the ratio of two integers) and are transcendental (can not be roots to polynomials with integer coefficients). These two numbers have a long history.

π is most famous for representing the ratio of the circumference C to the diameter d for any circle ($\pi = C/d$) and the approximation 3.16 was known since the time of ancient Egypt over 4000 years ago. Since then, this number (and its approximation) has appeared in many different mathematical contexts. The first theoretical calculation seems to have been carried out by Archimedes of Syracuse (287-212 BC). He obtained the approximation $^{223}/_{71} < \pi < ^{22}/_{7}$.

e is most famous for representing the base of the natural (or Napierian) logarithm ($\ln e = 1$); John Napier had referred to e (indirectly) in 1618. However, the number e has formally been discovered in connection with the problem of compounded interest in the form of a limit:

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

The first approximation of the above limit was by John Bernoulli (1683) who gave the lower and upper bounds 2 and 3, respectively. Euler (1748) coined the symbol "e" as the number representing the limit value. He also approximated e to 18 decimal places (likely using a power series approximation as shown below). One important observation about e is in the context of the exponential function $f(x) = e^x$ where the derivative of e^x with respect to x is e^x itself. This mathematical property plays an important role in the solution of ordinary linear differential equations. Also, in modern mathematics, e^x is the inverse of the natural logarithm function; i.e., if $g(x) = \ln x$, then $g^{-1}(x) = e^x$. We may write $g^{-1}(\ln x) = e^{\ln x} = x$ and $\ln e^x = x \ln e = x$.

The imaginary number $\sqrt{-1}$, commonly represented by the "imaginary unit" i was first encountered in connection with finding the roots of some third degree polynomials. i forms the basis for a complex number (the sum of a real and imaginary numbers). i plays an important role in Algebra, in Complex Analysis and in solving problems in many areas of science and engineering, including electrical engineering. For instance, according to the Fundamental Theorem of Algebra, an nth degree polynomial with real coefficients has exactly n complex roots, some of which could be repeated roots (it should be noted here that real numbers are special cases of complex numbers where the imaginary part is equal to zero). A short history of $\sqrt{-1}$ can be found here.

In the following, we explore the interconnected mathematical world of the above three numbers. The Maclaurin series, considered next, will provide us with a powerful tool to unlock some "beautiful" mathematical results involving *π*, *e* and $\sqrt{-1}$.

Taylor/Maclaurin Series

A **Taylor series** is a representation of a function as an infinite sum of terms that are calculated from the values of the function's derivatives at a single point.

The Taylor series of a real or complex function *f*(*x*) that is infinitely differentiable in a neighborhood of a real or complex number *a* is the power series:

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + ...$$

Here, $f'(a)$ means the derivative of *f*(*x*) evaluated at *x* = *a*, $f''(a)$ means the second derivative if *f*(*x*) evaluated at the point *a*, as so on.

The concept of a Taylor series was formally introduced by the English mathematician Brook Taylor in 1715. If the Taylor series is centered at zero, then that series is also called a **Maclaurin series** (named after the Scottish mathematician Colin Maclaurin, who made extensive use of this special case of Taylor series in the 18th century):

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + ...$$

Examples of Maclaurin (power) series (series converge for all values of *x* except where indicated):

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ... \quad |x| \le 1$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... \quad -1 < x \le 1$$

$$\ln\frac{1+x}{1-x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + ... \right) \quad |x| < 1$$

$$(1+x)^n = 1 + \frac{n}{1!}x + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ...$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 ...... \quad |x| < 1$$

It should be noted here that a number of power series have been known through different means of derivation before the time of Taylor and Maclaurin. For instance, the e^x , sin(x) and cos(x) power series were published by Newton in 1665-1666 and the power series for ln(1+x) was published in 1668 by Mercator.

Application of the Maclaurin Series

In the following sections, the above power series will be employed to (1) find the sum of convergent infinite series, (2) derive interesting infinite sum representations for π and e, (3) solve limit problems, (4) solve difficult integration problems, (5) derive Euler's identity, (6) derive trigonometric identities, and (7) provide other applications of complex numbers.

Examples of Divergent and Convergent Series

Finding the sum of an infinite series of numbers is a classic application of power series. In some cases, the sum does not converge to a finite number and the series is labeled "divergent." A classic example of a divergent series (sum equals infinity) is the Harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + ... = \infty$$

The divergence of the series is not obvious since the series diverges very, very slowly; the sum of the first thousand terms is approximately 7.485; for the 1 million terms it is approximately 14.393; for the 10^9 terms it is close to 21.300; and so on. Therefore proving that the sum of the series if infinity using addition is impossible (without using computers!). Nicolae Oresme (1323–1382), a French mathematician, proved the divergence of the Harmonic series by providing an upper bound in the form of an infinite sum that can be easily seen to go to infinity:

$$\begin{aligned} &1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} ...... \\ &= 1 + (\frac{1}{2} + \frac{1}{3}) + (\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}) + ... \\ &> 1 + (\frac{1}{2} + \frac{1}{2}) + (\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}) + ... = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + ... = \infty \end{aligned}$$

One example of a convergent infinite sum is the series (note that we are also referring to the sum of the series as just "series")

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+.....$$

We can show that this sum is 2 by simply noting that a circle with area 1 can be divided in half (area is 1/2 + 1/2 = 1), then one of the half's is divided by half [area is 1/2 + (1/4+1/4) = 1], then one of the 1/4 parts is divided in half [area is now 1/2 + 1/4 + (1/8+1/8) = 1], and so on till we get

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+.....=1$$

Therefore, the original series sums to 2 and we arrive at

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+.....=2$$

Now, we turn our attention to the Maclaurin series listed above and generate some convergent infinite series and their corresponding sums. Setting $x = ½$ in the power series for $1/(1- x)$, we get

$$\frac{1}{1-\frac{1}{2}}=2=1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+......\quad (note: \left|x=\frac{1}{2}\right|<1)$$

This is the series that we have just summed in the previous example!

Series Approximation of *e*

Now consider the power series for the exponential function and set $x = 1$. This leads to an infinite sum of fractions representation of e:

$$e=1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+......$$

Note that the nth term ($1/n!$) converges very quickly to zero as n increases. This property allows a good approximation of e using only the first few terms; here, the sum of the first 10 terms is (note: 0! = 1)

$$\sum_{n=0}^{9}\frac{1}{n!}=2.718281525573$$

This result agrees with e up to the first six digits to the right of the decimal point. Now consider the power series for $(1+x)^n$ and substitute x/n for x. This leads to the power series:

$$\left(1+\frac{x}{n}\right)^n = 1+\frac{n}{1!}\frac{x}{n}+\frac{n(n-1)}{2!}\frac{x}{n^2}^2+\frac{n(n-1)(n-2)}{3!}\frac{x}{n^3}^3+......$$

Performing the limit of the above equation as n tends to infinity leads to (note: $n(n\text{-}1)/n^2$ = 1-1/n = 1 for large n, etc) :

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = 1+\frac{x}{1!}+\frac{x}{2!}^2+\frac{x}{3!}^3+......$$

And since the right hand side is just the power series for e^x, we then obtain the expression for e as a limit (Euler 1748):

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

Setting x = 1 in the power series for ln(1+x) gives an infinite alternating series sum representation for ln(2):

$$\ln(2) = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+......$$

The convergence is slow, as evidenced by comparing the following calculations (the answers are only displayed for the first eight digits of the results):

$$\sum_{n=0}^{999}\frac{(-1)^n}{n+1} = 0.69264743 \qquad \ln(2) = 0.69314718$$

Series Approximation of π

An infinite series representation for π (first found by Leibniz in 1674) can be obtained by letting x = 1 in the power series for arctan(x) or $\tan^{-1}(x)$:

$$\tan^{-1}1 = \frac{\pi}{4} = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+......$$

Unfortunately, this series converges very slowly making it undesirable as a numerical method for approximating π, as evidenced by adding the first million terms in this series:

$$4 \cdot \sum_{n=0}^{10^6} \frac{(-1)^n}{2n+1} = 3.14159365 \qquad \pi = 3.14159265$$

In 1736, Euler derived the following series for π:

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots$$

Still, Euler's series must include a very large number of terms to arrive at a good approximation of π:

$$\sqrt{6 \cdot \sum_{n=1}^{10^6} \frac{1}{n^2}} = 3.1415917$$

Here is how Euler derived his series. Euler started with the power series for sin(x), divided it by x ($x \neq 0$) and substituted $x^2 = y$ (note that $y \neq 0$) and set the series to zero. This gave rise to the infinite degree polynomial:

$$1 - \frac{y}{3!} + \frac{y^2}{5!} - \frac{y^3}{7!} + \ldots\ldots = 0$$

He then solved for roots of sin(x) = 0 which are $x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots\ldots$ Now, the square of these roots (excluding zero) are the roots for the polynomial in y, namely: π^2, $(2\pi)^2$, $(3\pi)^2$, …. Euler knew form the theory of nth degree polynomials that the absolute value of the negative coefficient of the linear term (i.e., +1/3!) is the sum of the reciprocal roots of the polynomial, and hence

$$\frac{1}{3!} = \frac{1}{\pi^2} + \frac{1}{(2\pi)^2} + \frac{1}{(3\pi)^2} + \ldots\ldots$$

or

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots$$

Euler also derived other infinite sums for $\pi^2/8$, $\pi^2/12$, π^4, up to π^{26}. These results are given in Euler's famous book "Introductio in Analysin Infinitorum" (1748). The symbols $\pi, e, i, \sum, \int, f(x)$ all appear in Euler's writings.

Next, we derive a faster converging series for approximating π based on the sum of two inverse tangent functions and their power series expansion. Consider the trigonometric identity (which we will derive later, along with other common trigonometric identities, by employing complex algebra):

$$\tan^{-1}(\alpha) + \tan^{-1}(\beta) = \tan^{-1}\left(\frac{\alpha+\beta}{1-\alpha\beta}\right)$$

Now, the choice $\alpha = 1/2$ and $\beta = 1/3$ leads to the identity:

$$\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

By employing the power series for the inverse tangent function for $x = 1/2$ and $x = 1/3$, we obtain the following infinite series for π:

$$\frac{\pi}{4} = \left[\frac{1}{2} - \frac{\left(\frac{1}{2}\right)^3}{3} + \frac{\left(\frac{1}{2}\right)^5}{5} - \frac{\left(\frac{1}{2}\right)^7}{7} + \ldots\ldots\right] + \left[\frac{1}{3} - \frac{\left(\frac{1}{3}\right)^3}{3} + \frac{\left(\frac{1}{3}\right)^5}{5} - \frac{\left(\frac{1}{3}\right)^7}{7} + \ldots\ldots\right]$$

The first 10 terms of the above series result in an approximation for π that is accurate to the first six digits to the right of the decimal point:

$$4 \cdot \sum_{n=0}^{9} \frac{(-1)^n \cdot \left(\frac{1}{2}\right)^{2n+1}}{2n+1} + 4 \cdot \sum_{n=0}^{9} \frac{(-1)^n \cdot \left(\frac{1}{3}\right)^{2n+1}}{2n+1} = 3.141592579606$$

In 1706, John Machin, professor of astronomy in London, derived the identity:

$$\frac{\pi}{4} = 4\tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right)$$

From which he obtained an approximation to π that is accurate to 100 decimal places. He did it by substituting the power series for the inverse tangent function in the above equation, namely:

$$\frac{\pi}{4} = 4\left[\frac{1}{5} - \frac{1}{3(5^3)} + \frac{1}{5(5^5)} - \ldots\ldots\right] - \left[\frac{1}{239} - \frac{1}{3(239^3)} + \frac{1}{5(239^5)} - \ldots\ldots\right]$$

Classic Integrals That Lead to π

Consider the indefinite integral and its solution as listed in any standard table of integrals (C is a constant of integration)

$$\int \frac{dx}{1+x^2} = \tan^{-1} x + C$$

As a historical note, this integral was employed in 1671 by James Gregory to obtain the power series representation for the inverse tangent function before the advent of the Maclaurin series. He simply performed long division of the integrand. But it was Leibniz in 1674 who substituted 1 for x in this Gregory's power series to arrive at the first infinite series ever found for π.

Now, integrating from zero to one leads to:

$$\int_0^1 \frac{dx}{1+x^2} = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4}$$

Therefore, the area under the curve $1/(1+x^2)$ in the interval (0,1) is $\pi/4$. It can also be easily shown (employing the last result) that:

$$\int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2} \text{ and } \int_{-\infty}^\infty \frac{dx}{1+x^2} = \pi$$

Next, consider the indefinite integral $\int e^{\frac{-x^2}{2}} dx$. This integral cannot be expressed in terms of elementary functions (polynomials, trigonometric functions, exponentials and their inverses). This integral (in definite form) appears in the theory of probability. The power series of the exponential function can be used to arrive at a power series solution, as follows. Replacing x by $-x^2/2$ in the power series for e^x gives (after simplification)

$$e^{\frac{-x^2}{2}} = 1 - \frac{x^2}{2.1!} + \frac{x^4}{4.2!} - \frac{x^6}{8.3!} + \frac{x^8}{16.4!} - \ldots\ldots$$

Integrating the above equation gives the power series solution:

$$\int e^{\frac{-x^2}{2}} dx = \int \left(1 - \frac{x^2}{2.1!} + \frac{x^4}{4.2!} - \frac{x^6}{8.3!} + \frac{x^8}{16.4!} - \ldots\ldots\right) dx = x - \frac{x^3}{3.2.1!} + \frac{x^5}{5.4.2!} - \frac{x^7}{7.8.3!} + \frac{x^9}{9.16.4!} - \ldots\ldots + C$$

or

$$\int e^{\frac{-x^2}{2}} dx = x - \frac{x^3}{6} + \frac{x^5}{40} - \frac{x^7}{336} + \frac{x^9}{3456} - \ldots\ldots + C$$

Now we may approximate the definite integral $\int_0^a e^{\frac{-x^2}{2}}dx$ using the sum

$$a-\frac{a^3}{6}+\frac{a^5}{40}-\frac{a^7}{336}+\frac{a^9}{3456}-......$$

The sum of the first five terms leads to good accuracy (for practical problems) as long as $0 \le a \le 1$. Here, the approximation for a = 1 is 0.855646 (compare to 0.855624 obtained by numerical integration).

It is interesting to note that the above series converges for all a > 1. Integrating $e^{\frac{-x^2}{2}}$ from zero to infinity results in the Gaussian integral (named after Carl Gauss, also known as the Euler-Poisson integral. This result was also known to De Moivre since 1733):

$$\int_0^\infty e^{\frac{-x^2}{2}}dx=\frac{\sqrt{\pi}}{2}$$

And because of the even symmetry in $e^{\frac{-x^2}{2}}$, we may write

$$\int_{-\infty}^\infty e^{\frac{-x^2}{2}}dx=\sqrt{\pi}$$

Since the area under the curve $1/(1+x^2)$ over all x is π, we may write:

$$\int_{-\infty}^\infty \frac{dx}{1+x^2}=\left(\int_{-\infty}^\infty e^{\frac{-x^2}{2}}dx\right)^2$$

Here are three additional integrals for which the reader may want to approximate (for finite limits) employing power series:

$$\int_0^\infty \cos x^2 dx=\frac{1}{2}\sqrt{\frac{\pi}{2}}$$

$$\int_0^\infty \frac{\sin x}{x}dx=\frac{\pi}{2}$$

$$\int_{-\infty}^\infty \frac{\cos x}{1+x^2}dx=\frac{\pi}{e}$$

Power Series Method for the Evaluation of Limits

Consider the problem of finding the limit as x approaches zero for the function $\sin(x)/x$. Evaluating at x = 0 leads to 0/0, so we may apply L'Hospital's rule (taught in standard calculus classes) whereby the original limit is equal to the limit of the ratio of the derivative of the numerator term and the derivative of the denominator term. Thus,

$$\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{\cos x}{1}=1$$

Alternatively, we may find the limit by starting with the power series of sin(*x*), dividing it by *x* and then taking the limit as *x* approaches zero:

$$\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\left(\frac{x-\frac{x^3}{3!}+\frac{x^5}{5!}-.....}{x}\right)=\lim_{x\to 0}\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-.....\right)=1$$

Other Interesting Expressions and Problems Involving *e* and *π*

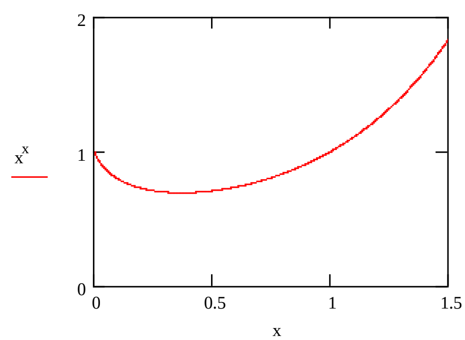
Sterling's Approximation of large factorials is an interesting approximation that relates π to *e*:

$$\lim_{n\to\infty}\frac{n!e^n}{\sqrt{n}n^n}=\sqrt{2\pi}$$

Or, for large *n*,

$$n!\approx\sqrt{2\pi n}n^n e^{-n}$$

The following are two extreme-value problems that involve *e* in their answer. Consider the function $f(x)=x^x$, for $x > 0$. Find the minimum value of *f*(*x*). Here is a plot for *f*(*x*):



Differentiating ln[*f*(*x*)] = *x* ln(*x*), we obtain:

$$\frac{d\ln f(x)}{dx}=\frac{1}{f(x)}\frac{d\,f(x)}{dx}=\frac{1}{x^x}\frac{d\,f(x)}{dx}$$

for the left hand side. And we obtain

$$\frac{d\, x \ln x}{dx} = 1 + \ln x$$

for the right hand side. Now, equating both sides, we obtain:

$$\frac{d\ f(x)}{dx} = x^x (1 + \ln x)$$

Setting the derivative to zero allow us to solve for the extreme points of the function. Thus, the minimum occurs at $x = 1/e$ and it has the value $f(1/e) = (1/e)^{(1/e)} \approx 0.6922006$. Note that this function has the value of 0^0 at $x = 0$. Next, we show that $\lim_{x\to 0} x^x = 1$. First, we note that

$$\lim_{x\to 0} \ln f(x) = \lim_{x\to 0} \ln x^x = \lim_{x\to 0} x \ln x = \lim_{x\to 0} \frac{\ln x}{1/x}$$

Since $\ln x/(1/x)$ at $x = 0$ has the value $-\infty/\infty$, we may apply L'Hospital differentiation rule to find the limit:

$$\lim_{x\to 0} \ln f(x) = \lim_{x\to 0} \frac{1/x}{-1/x^2} = \lim_{x\to 0} (-x) = 0$$

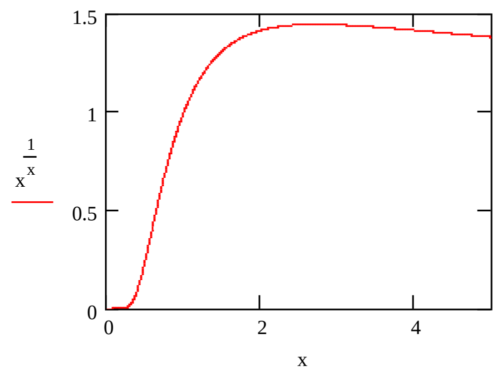
So,

$$\lim_{x\to 0} \ln x^x = \ln\left(\lim_{x\to 0} x^x\right) = 0$$

or [using ln(1) = 0],

$$\lim_{x\to 0} x^x = 1$$

Next, consider the function $f(x) = \sqrt[x]{x} = x^{1/x}$, for $x > 0$. Find the maximum value of $f(x)$. Here is a plot for $f(x)$:



Differentiating $f(x)$ and setting it equal to zero we obtain:

$$\frac{d\sqrt[x]{x}}{dx} = \sqrt[x]{x}\left(\frac{1 - \ln x}{x^2}\right) = 0$$

Thus, the minimum occurs at the point $x = e$ and it has the value $f(e) = (e)^{(1/e)} \approx 1.4446679$.

Note that at $x = 0$, this function has the value of $\sqrt[\infty]{\infty} = \infty^{\frac{1}{\infty}} = \infty^0$, and the limit as x approaches zero is undefined for this function. On the other hand, it can be shown that $\lim_{x\to\infty} \sqrt[x]{x} = 1$.

$\sqrt{-1}$ and Complex Numbers

$\sqrt{-1}$ appears in many contexts in mathematics. For instance, when solving for the roots of the polynomial $x^2 + 1 = 0$, we obtain $\pm\sqrt{-1}$. Euler (1707-1783) introduced the notation $i = \sqrt{-1}$ as the imaginary unit. In fact, a more proper definition for i is $i^2 = -1$. In electrical engineering, the symbol j is used as not to confuse i with the symbol for electric current.

A complex number z is a number $z = a + ib$, where a and b are real numbers. If a polynomial, with real coefficients, admits $z = a + ib$ as a root, then the complex conjugate, $z^* = a - ib$, must also be a root. For example, the cubic equation $x^3-3x^2+4x-2=0$ has the roots: 1, $1 + i$ and $1 - i$.

In 1797, Casper Wessel interpreted a complex number as a point in the *complex plane*, where the x-axis represented the real part of the number and the y-axis represented the complex part. Written in this form, the complex number is said to be in *rectangular* or *Cartesian* form. Wessel also gave an alternative representation of a complex number (*polar form*) as a radius vector (originating from the origin) having length (magnitude) r and angle θ(measured counter-clockwise from the x-axis to the radius vector. Thus, if $z = a + ib$, then its polar form is $r\angle\theta$, with

$$r = \sqrt{a^2 + b^2} \;\; and \;\; \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

or

$$a + ib = r\angle\theta = r\cos\theta + i(r\sin\theta)$$

When equating two complex quantities $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, we must then have the real parts equal to each other ($a_1 = a_2$) and the imaginary parts equal to each other ($b_1 = b_2$).

Since using a calculator to obtain θ returns the *principal value* (a result between $-\pi$ and $+\pi$) of the inverse tangent function, we must add π to the calculator result when $a < 0$. This is so because when a is negative, the complex number lies in the second or third quadrant.

For example, $1-i=\sqrt{1^2+(-1)^2}\angle\tan^{-1}\left(\frac{-1}{1}\right)=\sqrt{2}\angle-\frac{\pi}{4}$ on the other hand, $-1+i=\sqrt{(-1)^2+1^2}\angle\tan^{-1}\left(\frac{1}{-1}\right)=\sqrt{2}\angle-\frac{\pi}{4}+\pi=\sqrt{2}\angle\frac{3\pi}{4}$.

Complex number addition/subtraction is performed as follows:

$$a_1+ib_1\pm(a_2+ib_2)=(a_1+a_2)\pm i(b_1+b_2)$$

Two complex numbers are multiplied as follows:

$$(a_1+ib_1)(a_2+ib_2)=a_1a_2+ia_1b_2+ib_1a_2+i^2b_1b_2=(a_1a_2-b_1b_2)+i(a_1b_2+b_1a_2)$$

where $i^2=-1$ was used. Using the above result, it is easy to show that the product of two complex conjugate numbers $z.z^*$ is given by:

$$z.z^*=(a+ib)(a-ib)=a^2+b^2$$

Connections Between π, e and i

The Taylor/Maclaurin series listed above, are also valid for complex x. This allows us to derive a very useful representation for a complex number known as the *exponential form*. Substituting ix for x in the power series for the exponential function, gives:

$$e^{ix}=1+\frac{ix}{1!}+\frac{(ix)^2}{2!}+\frac{(ix)^3}{3!}+\frac{(ix)^4}{4!}+\frac{(ix)^5}{5!}......$$

Since i^2 = -1, i^3 = -i, i^4 = 1, i^5 = i, etc. we may rewrite the above series as

$$e^{ix}=1+\frac{ix}{1!}+\frac{i^2x^2}{2!}+\frac{i^3x^3}{3!}+\frac{i^4x^4}{4!}+\frac{i^5x^5}{5!}......=1-\frac{x^2}{2!}+\frac{x^4}{4!}+......+i\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-......\right)$$

From the above series, Euler (in 1746), observed that e^{ix} is the sum of the cos(x) power series and the sin(x) series multiplied by i; he wrote the identity (today known as Euler's identity):

$$e^{ix}=\cos x+i\sin x$$

[Also, note that: $e^{-ix}=\cos(-x)+i\sin(-x)=\cos x-i\sin x$]

Hence, the polar form $r\angle\theta$ of a complex number is just a symbolic representation of the analytic expression $re^{i\theta}$, $r\angle\theta$ = $re^{i\theta}$ = $r\cos(\theta)$ + i $r\sin(\theta)$. Euler identity results in what some mathematicians believe is the most "beautiful" mathematical expression (one that combines π, e and i):

$$e^{i\pi} = -1$$

Euler identity also leads to the following useful representations of the sine and cosine functions:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$
$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Interesting Formulas and Evaluations Based on Euler's Identity

Euler's identity is very useful in performing complex number multiplication and division. It is also a powerful tool in deriving many trigonometric identities, as is illustrated in this section.

Let $z = a + ib = r\angle\theta = e^{i\theta}$, $z_1 = r_1\angle\theta_1$ and $z_2 = r_2\angle\theta_2$, and let us find simplified expressions for z^*, z_1z_2, $z.z^*$, z_1/z_2, $1/z$, z^n, $z_1^{z_2}$, nth roots of unity (i.e., complex values of $1^{1/n}$) and $\sin^{-1}(2)$.

Complex conjugate:

$$z^* = (r\angle-\theta)^* = r\cos\theta - ir\sin\theta = r\cos(-\theta) + ir\sin(-\theta) = re^{-i\theta} = r\angle-\theta$$

Multiplication:

$$z_1z_2 = (r_1\angle\theta_1)(r_2\angle\theta_2) = r_1e^{i\theta_1}r_2e^{i\theta_2} = r_1r_2e^{i\theta_1+i\theta_2} = r_1r_2e^{i(\theta_1+\theta_2)} = r_1r_2\angle(\theta_1+\theta_2)$$

And as a special case,

$$zz^* = (r\angle\theta)(r\angle-\theta) = r^2\angle(\theta-\theta) = r^2\angle 0$$

Division:

$$\frac{z_1}{z_2} = \frac{r_1\angle\theta_1}{r_2\angle\theta_2} = \frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}} = \frac{r_1}{r_2}e^{i\theta_1}e^{-i\theta_2} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)} = \frac{r_1}{r_2}\angle(\theta_1-\theta_2)$$

And as a special case,

$$\frac{1}{z} = \frac{1\angle 0}{r\angle\theta} = \frac{1}{r}\angle-\theta$$

For example, if $z = i = e^{i\pi/2}$, we have $1/i = 1/e^{i\pi/2} = e^{-i\pi/2} = \cos(-\pi/2) + i\sin(-\pi/2) = 0 + i(-1) = -i$.

As another example, let us simplify the ratio $(1 + i)/(1 - i)$ by multiplying and dividing by $1 - i$, to obtain

$$\frac{1+i}{1-i} = \left(\frac{1+i}{1-i}\right)\left(\frac{1-i}{1-i}\right) = \frac{2}{-2i} = \frac{1}{-i} = i$$

<u>z raised to the power n :</u>

$$z^n = (r\angle\theta)^n = \left(re^{i\theta}\right)^n = r^n e^{in\theta} = r^n \angle n\theta$$

This result suggests that

$$(r\cos\theta + ir\sin\theta)^n = r^n\cos(n\theta) + ir^n\sin(n\theta)$$

or, by setting $r = 1$:

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

This last identity was known to Abraham De Moivre (1667-1754) which he had derived employing other means.

<u>Complex number raised to a complex number ($z_1^{z_2}$):</u>

Euler showed that $z_1^{z_2}$, where z_1 and z_2 are complex numbers, is a complex number. Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$. Note that z_1 may also be expressed as $r_1 e^{i(\theta_1 + 2k\pi)}$, where k is any integer ($k = 0, \pm 1, \pm 2, ...$) So, we may write (recall that $e^{\ln(a)} = a$ and $\ln(a^b) = b\ln a$):

$$z_1^{z_2} = e^{\ln z_1^{z_2}} = e^{z_2 \ln z_1} = e^{r_2 e^{i\theta_2} \ln\left(r_1 e^{i(\theta_1 + 2\pi k)}\right)} = e^{r_2 e^{i\theta_2} \ln r_1 + r_2 e^{i\theta_2}[i(\theta_1 + 2k\pi)]} = e^{r_2 \ln r_1 e^{i\theta_2} + ir_2(\theta_1 + 2k\pi)e^{i\theta_2}}$$
$$= e^{r_2 \ln r_1 (\cos\theta_2 + i\sin\theta_2) + ir_2(\theta_1 + 2k\pi)(\cos\theta_2 + i\sin\theta_2)} = e^{r_2[\ln r_1 \cos\theta_2 - (\theta_1 + 2k\pi)\sin\theta_2] + ir_2[\ln r_1 \sin\theta_2 + (\theta_1 + 2k\pi)\cos\theta_2]}$$

or,

$$(r_1\angle\theta_1)^{r_2\angle\theta_2} = \left(r_1 e^{i\theta_1}\right)^{r_2 e^{i\theta_2}} = e^{r_2[\ln r_1 \cos\theta_2 - (\theta_1 + 2k\pi)\sin\theta_2]} e^{ir_2[\ln r_1 \sin\theta_2 + (\theta_1 + 2k\pi)\cos\theta_2]}$$

This last result can now be used to find values for 1^i, i^i, $(1+i)^{(1+i)}$, and 1^π.

Find the value(s) of 1^i: Since $1 = 1e^{i0}$ and $i = 1e^{i\pi/2}$, then

$$1^i = e^{\ln(1)\cos\frac{\pi}{2} - (0+2k\pi).\sin\frac{\pi}{2}} e^{i\left[\ln(1)\sin\frac{\pi}{2} + (0+2k\pi)\cos\frac{\pi}{2}\right]} = e^{-2k\pi}e^0 = e^{-2k\pi}$$

For $k = 0$, we obtain the *principal value* $1^i = 1$; for $k = \pm 1$, we obtain $e^{\mp 2\pi}$; and so on.

Find the value(s) of i^i: Since $i = 1e^{i\pi/2}$, then

$$i^i = e^{(1)\left[\ln(1)\cos\frac{\pi}{2} - (\frac{\pi}{2}+2k\pi)\sin\frac{\pi}{2}\right]} e^{i\left[\ln(1)\sin\frac{\pi}{2} + (\frac{\pi}{2}+2k\pi)\cos\frac{\pi}{2}\right]} = e^{-\pi\left(\frac{1}{2}+2k\right)},\ k = 0, \pm1, \pm2, ...$$

For k = 0, we obtain the *principal value* $i^i \approx 0.2078796$.

Find the value(s) of $(1+i)^{(1+i)}$: Since $1+i = \sqrt{2}e^{i\frac{\pi}{4}}$, then

$$(1+i)^{(1+i)} = e^{\sqrt{2}\left[\ln\sqrt{2}\cos\frac{\pi}{4} - (\frac{\pi}{4}+2k\pi)\sin\frac{\pi}{4}\right]} e^{i\sqrt{2}\left[\ln\sqrt{2}\sin\frac{\pi}{4} + (\frac{\pi}{4}+2k\pi)\cos\frac{\pi}{4}\right]}$$
$$= e^{\left[\ln\sqrt{2} - \pi(\frac{1}{4}+2k)\right]} e^{i\left[\ln\sqrt{2} + \pi(\frac{1}{4}+2k)\right]},\ k = 0, \pm1, \pm2, ...$$

For k = 0, we obtain the *principal value*

$$(1+i)^{(1+i)} = e^{\left[\ln\sqrt{2} - \frac{\pi}{4}\right]} e^{i\left[\ln\sqrt{2} + \frac{\pi}{4})\right]} = e^{-\frac{\pi}{4}}\sqrt{2}\left[\cos\left(\ln\sqrt{2} + \frac{\pi}{4}\right) + i\sin\left(\ln\sqrt{2} - \frac{\pi}{4}\right)\right]$$
$$\approx 0.2739572 + i0.5837008$$

Here is another example whose result is needed below. Find the principal value for $\left(\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right)^i$. Solution:

$$\left(1\angle\pm\frac{\pi}{3}\right)^{1\angle\frac{\pi}{2}} = e^{(1)\left[\ln(1)\cos\frac{\pi}{2} - (\pm\frac{\pi}{3})\sin\frac{\pi}{2}\right]} e^{i(1)\left[\ln(1)\sin\frac{\pi}{2} + (\pm\frac{\pi}{3})\cos\frac{\pi}{2}\right]} = e^{\mp\frac{\pi}{3}}$$

Find the value(s) of 1^π: Since $1 = 1e^{i0}$ and $\pi = \pi e^{i0}$, then

$$1^\pi = e^{\pi[\ln(1)\cos 0 - (0+2k\pi)\sin 0]} e^{i\pi[\ln(1)\sin 0 + (0+2k\pi)\cos 0]} = e^{i2k\pi^2}$$
$$= \cos(2\pi^2 k) + i\sin(2\pi^2 k)$$

For k = 0, we obtain the *principal value* $1^\pi = 1$; for k = 1, we obtain (approx) 0.6296817 + i0.7768532.

It is left to the reader to find values for 1^e, and $1^{\sqrt{2}}$.

<u>Roots of the polynomial $z^n - 1 = 0$:</u>

From the fundamental theorem of algebra, we know that this equation must have n roots (including complex ones that are conjugate pairs). Solving for z, we get: $z = 1^{1/n}$. But, since $1 = e^{i2\pi k}$ then $1^{1/n} = e^{i2\pi k/n}$. The solutions correspond to all integer values of k,$|k| < n$. For example, if n = 1, then k = 0 lead to the single real solution $z = e^0 = 1$. If n = 2, then k = $0, \pm 1$ leads to the real solutions $z = e^0 \, and \, e^{\pm\pi}$, or z = 1 and -1. If n = 3, k = $0, \pm 1, \pm 2$ leads to one real and two complex conjugate roots: $z = 1, e^{\pm\frac{2\pi}{3}} = 1, 1\angle\frac{2\pi}{3}, 1\angle\frac{-2\pi}{3}$. And so on. It can be easily checked that the roots are distributed uniformly on the unit circle (in the complex plane).

Solve for $\sin^{-1}(2)$:

Ler $\theta = \sin^{-1} z$. Then, $z = \sin\,\theta$, and we may write

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = z \;\; or \;\; \frac{e^{i\theta} - e^{-i\theta}}{2i} - z = 0$$

Multiplying the left hand side by $2ie^{i\theta}$, and calling $y = e^{i\theta}$, leads to the quadratic equation

$$y^2 - 2izy - 1 = 0$$

Whose roots are (given by the quadratic formula):

$$y = e^{i\theta} = \frac{2iz \pm \sqrt{-4z^2 + 4}}{2} = iz \pm \sqrt{1 - z^2} = iz \pm i\sqrt{z^2 - 1} = i\left(z \pm \sqrt{z^2 - 1}\right)$$

Taking the natural log of this equation, and multiplying by $-i$, leads to

$$\theta = -i\ln\left[i\left(z \pm \sqrt{z^2 - 1}\right)\right] = -i\ln i - i\ln\left(z \pm \sqrt{z^2 - 1}\right)$$
$$= -i\ln e^{i\frac{\pi}{2}} - i\ln\left(z \pm \sqrt{z^2 - 1}\right) = \frac{\pi}{2} - i\ln\left(z \pm \sqrt{z^2 - 1}\right)$$

So, when z > 1, the solution is a complex number with real part $\pi/2$ and imaginary part $-\ln\left(z \pm \sqrt{z^2 - 1}\right)$. For z = 2, we have $\sin^{-1} 2 = \pi/2 - i\ln\left(2 \pm \sqrt{3}\right)$. The formula also works for $z < 1$. For example, setting $z = ½$, gives the two valid solutions

$$\theta = \frac{\pi}{2} - i\ln\left(\frac{1}{2} \pm \sqrt{2^2 - 1}\right) = \frac{\pi}{2} - i\ln\left(\frac{1}{2} \pm \sqrt{\frac{1}{4} - 1}\right) = \frac{\pi}{2} - i\ln\left(\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right)$$
$$= \frac{\pi}{2} - \ln\left(\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right)^i = \frac{\pi}{2} - \ln\left(e^{\mp\frac{\pi}{3}}\right) = \frac{\pi}{2} \pm \frac{\pi}{3} = \frac{5\pi}{6}, \frac{\pi}{6} = \frac{-\pi}{6}, \frac{\pi}{6}$$

Trigonometric identities:

Starting from

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$
$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

and squaring both identities, we obtain

$$\cos^2\theta + \sin^2\theta = \left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)^2 + \left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)^2 = \frac{1}{4}\left[\left(e^{i2\theta} + e^{-2i\theta} + 2\right) - \left(e^{i2\theta} + e^{-2i\theta} - 2\right)\right] = 1$$

or, the trigonometric identity

$$\cos^2\theta + \sin^2\theta = 1$$

Starting from

$$1\angle\alpha = \cos\alpha + i\sin\alpha$$
$$1\angle\beta = \cos\beta + i\sin\beta$$

Multiplying the two numbers gives

$$1\angle(\alpha+\beta) = \cos(\alpha+\beta) + i\sin(\alpha+\beta) = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$$
$$or$$
$$\cos(\alpha+\beta) + i\sin(\alpha+\beta) = (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\cos\alpha\sin\beta + \sin\alpha\cos\beta)$$

Which upon equating the real parts to each other and the imaginary parts to each other, we immediately have the two trigonometric identities

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\sin(\alpha+\beta) = \cos\alpha\sin\beta + \sin\alpha\cos\beta$$

Next, if we start from the identity (derived above)

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

And set n = ½, we obtain

$$(\cos\theta + i\sin\theta)^{\frac{1}{2}} = \cos\left(\frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2}\right)$$

Squaring both sides, we get

$$\cos\theta + i\sin\theta = \cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right) + i2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$$

When the real parts are equated, we get

$$\cos\theta = \cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)$$

And, since $\cos^2 x + \sin^2 x = 1$, we can rewrite the above identity as

$$\cos\theta = 1 - 2\sin^2\left(\frac{\theta}{2}\right) \;\; or \;\; \sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{1-\cos\theta}{2}}$$

$$\cos\theta = 2\cos^2\left(\frac{\theta}{2}\right) - 1 \;\; or \;\; \cos\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+\cos\theta}{2}}$$

Similarly, equating the imaginary parts leads to

$$\sin\theta = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$$

As a final derivation, consider the Cartesian product

$$(\alpha + i)(\beta + i) = (\alpha\beta - 1) + i(\alpha + \beta)$$

In polar form, the product is

$$\left(\sqrt{\alpha^2+1}\angle\tan^{-1}\left(\frac{1}{\alpha}\right)\right)\left(\sqrt{\beta^2+1}\angle\tan^{-1}\left(\frac{1}{\beta}\right)\right) = \sqrt{(\alpha\beta-1)^2 + (\alpha+\beta)^2}\angle\tan^{-1}\left(\frac{\alpha+\beta}{\alpha\beta-1}\right)$$

or

$$\sqrt{\alpha^2+1}\sqrt{\beta^2+1}\angle\left[\tan^{-1}\left(\frac{1}{\alpha}\right) + \tan^{-1}\left(\frac{1}{\beta}\right)\right] = \sqrt{(\alpha\beta-1)^2 + (\alpha+\beta)^2}\angle\tan^{-1}\left(\frac{\alpha+\beta}{\alpha\beta-1}\right)$$

Which upon equating the angles, gives rise to the identity used earlier (see section on Series Approximation of π)

$$\tan^{-1}\left(\frac{1}{\alpha}\right) + \tan^{-1}\left(\frac{1}{\beta}\right) = \tan^{-1}\left(\frac{\alpha+\beta}{\alpha\beta-1}\right)$$

or

$$\tan^{-1}(\alpha) + \tan^{-1}(\beta) = \tan^{-1}\left(\frac{\alpha+\beta}{1-\alpha\beta}\right)$$

Application of Complex Numbers to Infinite Walks

Consider the problem of infinite walk in a plane. Assume a person starts at the origin and moves in the positive *x*-axis direction for one step. He then pivots on his heels in a counter-clock-wise (CCW) way through angle θ and takes half a step forward. He then pivots again through a CCW rotation of θ and takes a quarter of a step forward. He continues this walk pattern forever. At what angle θ is he furthest away (1) form the *x*-axis? (2) from the point he started at?

Employing polar coordinates, we may describe the location of the person in the complex plane by the following infinite series:

$$S(\theta) = 1 + \frac{1}{2}e^{i\theta} + \frac{1}{4}e^{i2\theta} + \frac{1}{8}e^{i3\theta} ..... = \left(\frac{1}{2}e^{i\theta}\right)^0 + \left(\frac{1}{2}e^{i\theta}\right)^1 + \left(\frac{1}{2}e^{i\theta}\right)^2 + \left(\frac{1}{2}e^{i\theta}\right)^3 .....$$

This is a geometric series because the ratio of any two consecutive terms is a constant, $(1/2)e^{i\theta}$. The series converges to (this is a standard result in any calculus book):

$$S(\theta) = \frac{1}{1 - \frac{1}{2}e^{i\theta}}$$

In rectangular form, we may express $S(\theta)$ as:

$$S(\theta) = S_r(\theta) + iS_i(\theta) = \frac{1}{1 - \frac{1}{2}e^{i\theta}} = \frac{1}{1 - \frac{1}{2}(\cos\theta + i\sin\theta)} = \frac{1}{\left(1 - \frac{1}{2}\cos\theta\right) - i\frac{1}{2}\sin\theta}$$

$$= \frac{1}{\left(1 - \frac{1}{2}\cos\theta\right) - i\frac{1}{2}\sin\theta} \, \frac{\left(1 - \frac{1}{2}\cos\theta\right) + i\frac{1}{2}\sin\theta}{\left(1 - \frac{1}{2}\cos\theta\right) + i\frac{1}{2}\sin\theta} = \frac{\left(1 - \frac{1}{2}\cos\theta\right) + i\frac{1}{2}\sin\theta}{\left(1 - \frac{1}{2}\cos\theta\right)^2 + \frac{1}{4}\sin^2\theta}$$

$$= \frac{\left(1 - \frac{1}{2}\cos\theta\right) + i\frac{1}{2}\sin\theta}{1 - \cos\theta + \frac{1}{4}\cos^2\theta + \frac{1}{4}\sin^2\theta} = \frac{\left(1 - \frac{1}{2}\cos\theta\right) + i\frac{1}{2}\sin\theta}{\frac{5}{4} - \cos\theta}$$

$$= \frac{\left(1 - \frac{1}{2}\cos\theta\right)}{\frac{5}{4} - \cos\theta} + i\frac{\frac{1}{2}\sin\theta}{\frac{5}{4} - \cos\theta}$$

Where S_r and S_i are the real and imaginary parts of S, respectively. For the person position to be furthest away from the x-axis, we must have $\frac{dS_i(\theta)}{d\theta} = 0$.

$$\frac{d}{d\theta}\left(\frac{\frac{1}{2}\sin\theta}{\frac{5}{4}-\cos\theta}\right) = -\frac{1}{2}\frac{1+\frac{5}{4}\cos\theta}{\left(\frac{5}{4}-\cos\theta\right)^2} = 0$$

Solving, we obtain $\theta = \cos^{-1}(-4/5) = 0.6435011$ or about 36.87°, and so the maximum value for S_i is 2/3; i.e., $S = 4/3 + i(2/3)$ is the solution point in the complex plane.

The distance from the origin, $D(\theta)$ is given by:

$$D(\theta) = \sqrt{S_r^{\,2}(\theta) + S_i^{\,2}(\theta)} = \frac{1}{\sqrt{\frac{5}{4}-\cos\theta}}$$

The angle that maximizes this distance is then (by inspection) $\theta = 0$ for which $D(0) = 2$. This corresponds to the convergent series that we have encountered earlier:

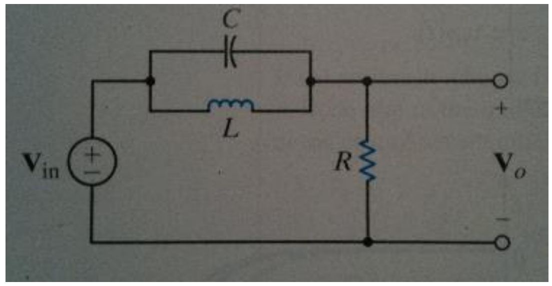
$$D(0) = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... = 2$$

On the other hand, the angle $\theta = \pi$ leads to the minimum distance $D(\pi) = 2/3$. This corresponds to the case where the person makes a step forward, then a half step backward, then a ¼ step forward, then 1/8 step backward, and so on; as described by the following series:

$$D(\pi) = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + ... = \frac{2}{3}$$

Application of Complex Numbers to Steady-State Circuit Analysis

Complex number algebra allows for a powerful, yet simple method for analyzing linear electric circuits operating in the steady-state mode with sinusoidal excitation. The method is known as the *phasor analysis* method. If the input voltage driving such circuits is of the form $V_{in}\cos(\omega t)$, then all circuit voltage and current responses are sinusoids oscillating at the same input frequency ω; but may differ in their amplitude and phase. The computation of such responses can be achieved through pure algebraic manipulation of expressions involving complex numbers (as opposed to solving differential equations). For example, consider the following circuit shown below.



The output voltage phasor representation $\mathbf{V}_o = V_o \angle \theta$ is given by:

$$\mathbf{V}_o = \mathbf{H}(\omega)\mathbf{V}_{in}$$

Where, $\mathbf{V}_{in} = V_{in} \angle 0$ and $\mathbf{H}(\omega) = H(\omega) \angle \theta(\omega)$ is the circuit's transfer function (a complex quantity that is a function of frequency) that is computed using circuit analysis techniques. Thus, the output voltage (in phasor form) is given by:

$$\mathbf{V}_o = V_o \angle \theta = [H(\omega) \angle \theta(\omega)] V_{in} \angle 0 = H(\omega) V_{in} \angle \theta(\omega)$$

And the time-domain output signal is simply, $v_o(t) = H(\omega)V_{in}\cos[\omega t + \theta(\omega)]$.

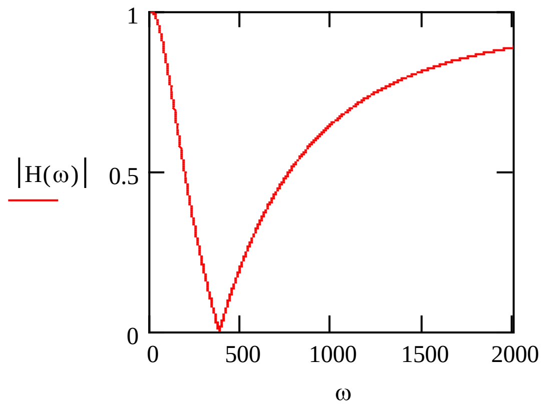
The above circuit has the transfer function:

$$\mathbf{H}(\omega) = \frac{R}{R + i\dfrac{\omega L}{1 - \omega^2 LC}}$$

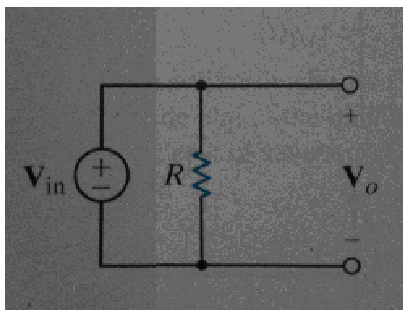
The output voltage amplitude is proportional to the magnitude $H(\omega)$ of this complex quantity. This particular circuit is that of a notch filter. It is designed to suppress a narrow range of frequencies. Let us now find the frequency that is most attenuated by this circuit. $H(\omega)$ is given by:

$$H(\omega) = \frac{R}{\sqrt{R^2 + \dfrac{\omega^2 L^2}{\left(1 - \omega^2 LC\right)^2}}}$$

The minimum value of $H(\omega)$ occurs when the denominator is maximized; i.e., when the term $\left(1 - \omega^2 LC\right)$ approaches zero. This leads to: $\omega = \dfrac{1}{\sqrt{LC}}$. Physically speaking, at this frequency, the impedance of the LC part of the circuit is very large; which prevents current from passing to the resistor. This, in turn, pushes the voltage $\mathbf{V}_o$, across the resistor, to zero (by Ohm's Law). A plot of $H(\omega)$, for $\dfrac{1}{\sqrt{LC}} = 120\pi$, is shown below.



The above plot suggests that for very low frequencies and very high frequencies $H(\omega)$ approaches 1 (no attenuation of input signal amplitude). Here, $v_o(t) = v_{in}(t)$. This can be explained qualitatively by noting that for $\omega = 0$, the capacitor behaves as an open circuit and the inductor behaves as a short circuit. Similarly, for ω approaching ∞, the capacitor behaves a s a short circuit and the inductor behaves as an open circuit. In both cases, the circuit reduces to the following one, which leads to $v_o(t) = v_{in}(t)$.



Exercise: Find the response $v_o(t)$ of the above circuit to the input $v_{in}(t) = \cos(1000\pi t)$. Assume, $R = 10$, $C = 10^{-4}$ and $L = 0.073$. Compare $v_o(t)$ to $v_{in}(t)$ by plotting.

Complex Numbers, Exponential Functions and Differential Equations

The exponential function of the form $e^{(a+ib)x}$ plays an important role in the solution of ordinary constant-coefficient homogeneous differential equations. In some instances, the constant real coefficient a (or b) can be zero. This is illustrated by three examples.

Consider the following first order differential equation:

$$\frac{dy(x)}{dx} + 2y(x) = 0, \text{ with initial condition } y(0) = -1.$$

We are interested in the solution $y(x)$, for $x > 0$. The solution must be a function $y(x)$ whose derivative is equal to $-2y(x)$. The exponential Ae^{-2x} satisfy this requirement, since:

$$\frac{dAe^{-2x}}{dx} + 2Ae^{-2x} = -2Ae^{-2x} + 2Ae^{-2x} = 0$$

The constant *A* is obtained by evaluating the solution at the initial condition: Ae^0 = -1 or *A* = -1. Hence, y(x) = $-e^{-2x}$.

Next, consider the second-order differential equation:

$$\frac{d^2y(x)}{dx^2} + 3\frac{dy(x)}{dx} + 2y(x) = 0, \text{ with } \frac{dy(0)}{dx} = -1 \text{ and } y(0) = 0$$

The solution is of the form $Ae^{-x} + Be^{-2x}$. The exponents -1 and -2 are the roots of the characteristic polynomial $s^2 + 3s + 2 = (s+1)(s+2) = 0$, obtained by starting with the above differential equation and substituting s^n for the *n*th derivative term [here, $y(x) = d^0y(x)/dx^0 = s^0 = 1$]. Here is a verification of the solution:

$$\begin{aligned}
&\frac{d^2\left(Ae^{-x} + Be^{-2x}\right)}{dx^2} + 3\frac{d\left(Ae^{-x} + Be^{-2x}\right)}{dx} + 2\left(Ae^{-x} + Be^{-2x}\right) \\
&= Ae^{-x} + 4Be^{-2x} + 3\left(-Ae^{-x} - 2Be^{-2x}\right) + 2\left(Ae^{-x} + Be^{-x}\right) \\
&= 0
\end{aligned}$$

The constants *A* and *B* are obtained from the initial conditions, as follows:

$$\begin{aligned}
&\frac{dy(0)}{dx} = -Ae^0 - 2Be^0 = A + 2B = 1 \\
&y(0) = Ae^0 + Be^0 = A + B = 0
\end{aligned}$$

The constants are the solution to the above system of two linear equations, and is given by: *A* = -1 and *B* = 1. Hence, $y(x) = e^{-2x} - e^{-x}$.

As a final example, consider the second order differential equation:

$$\frac{d^2y(x)}{dx^2} + 2\frac{dy(x)}{dx} + 2y(x) = 0, \text{ with } \frac{dy(0)}{dx} = 0 \text{ and } y(0) = -1$$

The roots of the characteristic polynomial $s^2 + 2s + 2 = 0$ are (employing the quadratic formula): $-1 + i$ and $-1 - i$. Hence, the solution is: $y(x) = Ae^{(-1+i)x} + Be^{(-1-i)x}$, which may be re-formulated (by expanding and then employing Euler's formula):

$$\begin{aligned}
&y(x) = e^{-x}\left(A\cos x + iA\sin x + B\cos x - iB\sin x\right) = e^{-x}\left(\left(A+B\right)\cos x + i(A-B)\sin x\right) \\
&= e^{-x}\left(C_1\cos x + C_2\sin x\right)
\end{aligned}$$

Employing the initial conditions, we obtain:

$$\frac{dy(0)}{dx} = -C_1 + C_2 = 0$$
$$y(0) = C_1 = -1$$

Therefore the solution is:

$$y(x) = -e^{-x}(\cos x + \sin x)$$

Complex Numbers and Difference Equations

Consider the difference equation:

$$u_{n+2} - 4u_{n+1} + 8u_n = 0, \text{ with } u_1 = u_0 = 1$$

which has values: 1, 1, -4, -24, -64, -64, …. , for n = 0, 1, 2, 3, 4, 5, ….

The characteristic polynomial for this difference equation is: $z^2 - 4z + 8$, which has the roots: $z_{1,2} = 2 \pm i2 = 2\sqrt{2}e^{\pm i\frac{\pi}{4}}$. The solution u_n is of the form:

$$u_n = A\left(2\sqrt{2}e^{i\frac{\pi}{4}}\right)^n + B\left(2\sqrt{2}e^{-i\frac{\pi}{4}}\right)^n$$

The constants A and B are determined from the initial conditions $u_1 = u_0 = 1$, by solving the two equations:

$$A + B = 1$$
$$2\sqrt{2}Ae^{i\frac{\pi}{4}} + 2\sqrt{2}Be^{-i\frac{\pi}{4}} = 1$$

Solving, we obtain: $A = \frac{1}{2} + i\frac{1}{4} = \frac{\sqrt{5}}{4}e^{i\tan^{-1}\left(\frac{1}{2}\right)}$ and $B = \frac{1}{2} - i\frac{1}{4} = \frac{\sqrt{5}}{4}e^{-i\tan^{-1}\left(\frac{1}{2}\right)}$. Therefore the solution to the difference equation is:

$$u_k = \frac{\sqrt{5}}{4}\left(2\sqrt{2}\right)^n\left\{e^{i\left[\frac{n\pi}{4}+\tan^{-1}\left(\frac{1}{2}\right)\right]} + e^{-i\left[\frac{n\pi}{4}+\tan^{-1}\left(\frac{1}{2}\right)\right]}\right\} = \frac{\sqrt{5}}{2}\left(2\sqrt{2}\right)^n\cos\left[\frac{n\pi}{4} + \tan^{-1}\left(\frac{1}{2}\right)\right]$$

Now, recalling the formula $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$, and observing that

$$\cos\left[\tan^{-1}\left(\frac{1}{2}\right)\right] = \frac{2}{\sqrt{5}}$$
$$\sin\left[\tan^{-1}\left(\frac{1}{2}\right)\right] = \frac{1}{\sqrt{5}}$$

It then follows that

$$u_k = \frac{1}{2}\left(2\sqrt{2}\right)^n\left[2\cos\left(\frac{n\pi}{4}\right) - \sin\frac{n\pi}{4}\right] = 2^{\frac{3n}{2}-1}\left[2\cos\left(\frac{n\pi}{4}\right) - \sin\frac{n\pi}{4}\right],\ n = 0,\ 1,\ 2,\ 3,\ldots.$$