The System Analysis Method (Continued)

Example: Analysis of a second-order LTIC system employing the system’s method. Find $y(t) = y_{zi}(t) + y_{zs}(t), \ t \geq 0$ for the system described by the second order linear differential equation

$$y''(t) + 5y'(t) + 6y(t) = f(t)$$

with

$$f(t) = 4e^{-t}u(t), \ y(0) = 10, \ y'(0) = -2$$

where $y' = \frac{dy}{dt}$.

Note: In Lecture 7 we solved this system for $y(t) = y_n + y_f$ and obtained

$$y(t) = \underbrace{24e^{-2t} - 16e^{-3t}}_{y_n} + \underbrace{2e^{-t}}_{y_f}, \ t > 0$$

Solution:

$$Q(D)y(t) = P(D)f(t)$$

$$Q(D) = D^2 + 5D + 6, \ P(D) = 1; \ n = 2, \ m = 0$$

$$y_{zi}(t) = ?$$

$$Q(s) = s^2 + 5s + 6 = 0$$
\[(s + 2)(s + 3) = 0 \rightarrow s_1 = -2, \ s_2 = -3\]

Natural modes: \[e^{-2t}, e^{-3t}\]

\[y_{zi}(t) = A_1 e^{-2t} + A_2 e^{-3t}, t \geq 0\]

Employ the I.C. to find \(A_1\) and \(A_2\) [Note: In the classic method we applied the I.C. to the complete response, \(y(t)\). Here, it is applied to \(y_{zi}(t)\)],

\[
\begin{align*}
y_{zi}(0) &= 10 = A_1 + A_2 \quad \rightarrow \quad A_1 = 28 \\
y'_{zi}(0) &= -2 = -2A_1 - 3A_2 \quad \rightarrow \quad A_2 = -18
\end{align*}
\]

Therefore,

\[y_{zi}(t) = (28e^{-2t} - 18e^{-3t})u(t)\]

Zero-state response:

\[y_{zs}(t) = f(t) \ast h(t), \text{ where } f(t) = 4e^{-t}u(t)\]

But we need \(h(t)\). The form of \(h(t)\) for a LTI system is given by

\[h(t) = [P(D)h_0(t)]u(t) + b\delta(t), \ (\text{with } b\delta(t) \rightarrow 0 \text{ because } m < n)\]

where \(h_0(t)\) is the sum of the natural modes:

\[h_0(t) = B_1 e^{-2t} + B_2 e^{-3t}\]

Employing the appropriate initial conditions for \(h_0(t)\) we obtain

\[
\begin{align*}
h_0(0) &= 0 \quad \rightarrow \quad 0 = B_1 + B_2 \\
h'_0(0) &= 1 \quad \rightarrow \quad 1 = -2B_1 - 3B_2
\end{align*}
\]

\[
\begin{align*}
&\rightarrow \quad B_1 = 1 \\
&\rightarrow \quad B_2 = -1
\end{align*}
\]

Therefore, \(h_0(t) = e^{-2t} - e^{-3t}\) and

\[h(t) = [P(D)h_0(t)]u(t) = [(1)(e^{-2t} - e^{-3t})]u(t) = (e^{-2t} - e^{-3t})u(t)\]
\[ y_{zs}(t) = f(t) \ast h(t) = 4e^{-t}u(t) \ast (e^{-2t} - e^{-3t})u(t) \]
\[ = 4e^{-t}u(t) \ast e^{-2t}u(t) - 4e^{-t}u(t) \ast e^{-3t}u(t) \]

Employ Convolution Table (Pair #4) to obtain,
\[ y_{zs}(t) = (-4e^{-2t} + 2e^{-3t} + 2e^{-t})u(t) \]

Therefore, the complete response is,
\[ y(t) = y_{zi} + y_{zs} \]
\[ = (28e^{-2t} - 18e^{-3t})u(t) + (-4e^{-2t} + 2e^{-3t} + 2e^{-t})u(t) \]

We may now employ the above result to determine (by inspection) the natural, forced, transient and steady-state responses as (why?),
\[ y_n(t) = (24e^{-2t} - 16e^{-3t})u(t); \quad y_f(t) = 2e^{-t}u(t) \]
\[ y_{tr}(t) = (24e^{-2t} - 16e^{-3t} + 2e^{-t})u(t); \quad y_{ss}(t) = 0 \]

**Note:** If we have the complete solution \( y_C(t) \) (say, obtained using the classical method) then the zero-state response can be found as
\[ y_{zs}(t) = y_C(t)u(t) - y_{zi}(t) \]

**Your turn:** Find \( y(t) = y_{zi}(t) + y_{zs}(t), \ t \geq 0 \), for the system described by the second-order linear differential equation
\[ \ddot{y}(t) + 3\dot{y}(t) + 2y(t) = \dot{f}(t) + 2f(t) \]

with
\[ f(t) = 4 \cos(t) \ u(t), \ y(0) = 1, \dot{y}(0) = 0 \]

Employ your solution to determine (by inspection) the natural, forced, transient and steady-state responses.

**Ans.** \( y_{zi}(t) = 2e^{-t}u(t) - e^{-2t}u(t); \ y_{zs}(t) = \left[ 2\sqrt{2} \cos \left( t - \frac{\pi}{4} \right) - 2e^{-t} \right] u(t) \)
**Example.** Find the steady-state current in the following circuit. Assume, $v_c(0^-) = 2$.

Employing capacitor voltage continuity condition: $v_c(0^+) = v_c(0^-) = 2$

$$y_{ss}(t) = i_{ss}(t) =?$$

$$y(t) = y_{zi}(t) + y_{zs}(t)$$

The circuit is passive, so it is stable and we have $y_{ss}(t) = \lim_{t \to \infty} y(t)$.

We have derived earlier (Lecture 9) the zero-input response [with $v_c(0^-) = 2$] and the unit-impulse response for this circuit and obtained:

$$y_{zi}(t) = -2e^{-t}u(t), \quad h(t) = \delta(t) - e^{-t}u(t)$$

We may employ convolution to find the zero-state response,

$$y_{zs}(t) = \cos(\omega t) u(t) \ast [\delta(t) - e^{-t}u(t)]$$

$$= \cos(\omega t) u(t) \ast \delta(t) - e^{-t}u(t) \ast \cos(\omega t) u(t)$$

$$= \cos(\omega t) u(t) - e^{-t}u(t) \ast \cos(\omega t) u(t)$$

Employing the following Convolution Table pair (with $\lambda = -1$)

$$\cos(\omega t) u(t) \ast e^{\lambda t} u(t) = \frac{1}{\lambda^2 + \omega^2} [\lambda e^{\lambda t} - \lambda \cos(\omega t) + \omega \sin(\omega t)] u(t)$$
leads to the complete response,

$$y(t) = -2e^{-t}u(t) + \cos(\omega t)u(t) - \frac{1}{1 + \omega^2}[-e^{-t} + \cos(\omega t) + \omega \sin(\omega t)]u(t)$$

$$y_{ss}(t) = \lim_{t \to \infty} y(t) \quad [\text{Note: } u(\infty) = 1]$$

$$= \cos(\omega t) - \frac{1}{1 + \omega^2}(\cos(\omega t) + \omega \sin(\omega t))$$

$$= \frac{\omega^2}{1 + \omega^2}\cos(\omega t) - \frac{\omega}{1 + \omega^2}\sin(\omega t)$$

Recall, \(a \cos(\omega t) + b \sin(\omega t) = \sqrt{a^2 + b^2} \cos(\omega t + \theta), \theta = \tan^{-1}\left(\frac{-b}{a}\right)\)

with \(a = \frac{\omega^2}{1 + \omega^2} > 0\). The steady-state response simplifies to,

$$y_{ss}(t) = \frac{\omega}{\sqrt{1 + \omega^2}}\cos\left[\omega t + \tan^{-1}\left(\frac{1}{\omega}\right)\right]$$

Amplitude: \(C = \frac{\omega}{\sqrt{1 + \omega^2}} \to \begin{cases} 0 & \text{as } \omega \to 0 \\ 1 & \text{as } \omega \to \infty \end{cases}\)

Angle/phase: \(\theta = \tan^{-1}\left(\frac{1}{\omega}\right) = \begin{cases} \frac{\pi}{2} & \text{as } \omega \to 0 \\ 0 & \text{as } \omega \to \infty \end{cases}\)
Since,

\[ \lim_{\omega \to 0} y_{ss}(t) = 0 \]
\[ \lim_{\omega \to \infty} y_{ss}(t) = \cos(\omega t) \text{ Amps} \]

the circuit is a high-pass filter: It attenuates low-frequency inputs and passes high-frequency inputs.

We could have predicted that the circuit blocks DC voltages by noting that the capacitor behaves as an open circuit under DC steady-state operation. Also, since the capacitor behaves as a short circuit for high frequencies, we can see that the (high-frequency) AC steady-state resistor current is simply,

\[ i(t) = \frac{f(t)}{R} = \frac{f(t)}{1\Omega} = \cos(\omega t) \text{ Amps} \]
Mathcad verification:

\[ yss(t, w) := \frac{w}{\sqrt{1 + w^2}} \cdot \cos \left( w \cdot t + \arctan \left( \frac{1}{w} \right) \right) \]

\[ w := \frac{1}{4} \]

\[ w := 1 \]

\[ w := 2 \]
Method #2: Phasor method for finding the steady-state response, $i_{ss}(t)$.

Phasor representation of the input voltage:

$$v_{in}(t) = (1) \cos(\omega t + 0) \rightarrow V_{in}(\omega) = 1\angle 0$$

KVL:

$$-1\angle 0 + \frac{1}{j\omega C} I + RI = -1\angle 0 + \frac{1}{j\omega} I + (1)I = 0$$

$$I = \frac{1\angle 0}{1 + \frac{1}{j\omega}} = \frac{1}{1 - j\frac{1}{\omega}}$$

Recall that, $\frac{1}{r\angle \theta} = \frac{1}{r} \angle -\theta$

$$I = \frac{1}{1 - j\frac{1}{\omega}} = \frac{1}{\sqrt{1 + \left(-\frac{1}{\omega}\right)^2}} \angle -\tan^{-1}\left(-\frac{1}{\omega}\right)$$

$$I = \frac{1}{\sqrt{1 + \frac{1}{\omega^2}}} \angle -\tan^{-1}\left(-\frac{1}{\omega}\right) = \frac{\omega}{\sqrt{1 + \omega^2}} \angle \tan^{-1}\left(\frac{1}{\omega}\right)$$

Transforming the phasor representation to the time-domain representation leads to,

$$i_{ss}(t) = \frac{\omega}{\sqrt{1 + \omega^2}} \cos \left[ \omega t + \tan^{-1}\left(\frac{1}{\omega}\right) \right]$$

which is the same result obtained in the previous example. Note the significant advantage of the phasor method for obtaining the steady-state analysis; there is no need for the complete response.
**Stability of LTIC Systems.** System stability is determined by the location of the system’s natural frequencies in the complex plane. The roots of the characteristic polynomial (characteristic roots) are the natural frequencies of the linear system,

\[ Q(s) = 0 \rightarrow \text{Roots: } s_i = \sigma_i \pm j\omega_i, \ i = 1, 2, ..., n \]

Assume \( s_i \neq s_j, \forall i, j \) (i.e., distinct roots) then the zero-input response is

\[ y_{zi}(t) = \sum_{i=1}^{n} A_i e^{s_it} = \sum_{i=1}^{n} A_i e^{\sigma_it} e^{\pm j\omega_it} \]

For (asymptotic) stability, we must have

\[ \lim_{t \to \infty} y_{zi}(t) = 0 \]

which requires all \( \sigma_i < 0 \). Therefore, a stable system must have all its natural frequencies in left hand side of the complex plane. A system is unstable if it has one (or more) natural frequency in the right hand side of the complex plane. The following figure depicts two examples:

A system is said to be marginally stable if it has natural frequencies on the \( j\omega \) axis (but no multiple frequencies on the \( j\omega \) axis) and no natural frequencies in the right hand side of the complex plane. See the following example of a second-order marginally stable system (\( s = \pm j\omega \)):
\[ y_{zi}(t) = A_1 e^{+j\omega t} + A_2 e^{-j\omega t} = Acos(\omega t + \theta) \]

The **transfer function** of a linear system is rational (ratio of two polynomials) and takes the form,

\[ H(s) = \frac{P(s)}{Q(s)} = \frac{K(s - z_1)(s - z_2) \ldots (s-z_m)}{(s - s_1)(s - s_2) \ldots (s - s_n)} \]

The natural frequencies are the roots of the denominator of \( H(s) \).

Therefore, the natural frequencies of a linear system are also referred to as the **poles** of the system [recall that they are the roots of \( Q(s) = 0 \)].

Equivalently, the stability of a linear system can be determined based on the unit-impulse response, \( h(t) \). It can be shown that a causal, continuous-time LTI system is stable if its \( h(t) \) is absolutely integrable, that is,

\[ \int_{0^-}^{\infty} |h(t)| dt < \infty \] (i.e., the integral has a finite value)

This condition can be verified by showing that (your turn: Why?) if the natural frequencies \( s = \sigma \pm j\omega \) are located in the LHP (including multiple frequencies) then the following integrals are finite,

\[ \int_{0^-}^{\infty} \delta(t) dt, \int_{0^-}^{\infty} |e^{\sigma t}| dt, \int_{0^-}^{\infty} |te^{\sigma t}| dt, \int_{0^-}^{\infty} |e^{\sigma t}\cos(\omega t)| dt, \int_{0^-}^{\infty} |te^{\sigma t}\cos(\omega t)| dt \]
Examples of Stable and Unstable Systems

Characterisic Root Location  Zero-Input Response  Characterisic Root Location  Zero-Input Response

(a)  

(b)  

(c)  

(d)  

(e)  

(f)
Notice how (refer to the above figure) the presence of overlapping (double) natural frequencies on the $j\omega$ axis lead to instability.

**Your turn:** Answer the questions for each of the following circuits.

1. Determine the order of the circuit by inspection.
2. Solve for the natural frequencies of the circuit. Is your answer in Part 1 correct?
3. Discuss the stability of the circuit. If the circuit is oscillatory, what is the oscillation frequency?
The Laplace Transform

The Laplace transform is a mathematical transformation that transforms integral-differential equations in time \((t)\) into complex algebraic equations in a complex frequency \((s)\) domain. Therefore, the Laplace transform plays an important role in simplifying the analysis of linear systems.

\[
\text{differential equation} \quad \longleftrightarrow \quad \text{Algebraic equation}
\]

When applied to a function \(f(t)\), the transform results in a complex rational function \(F(s)\).

\[
\text{(Time-domain)} \quad f(t) \leftrightarrow F(s) \quad \text{(s-domain)}
\]

Formal definition of the (one-sided or unilateral) Laplace transform:

\[
F(s) = L\{f(t)\} \triangleq \int_{0^-}^{\infty} f(t)e^{-st} \, dt
\]

where \(s = \sigma + j\omega\).

The Laplace transform has another integral that transforms back (inverse-transform) a rational \(s\)-domain function into the \(t\)-domain:

\[
L^{-1}\{F(s)\}
\]
Example. Find the Laplace transform for the unit-step function.

\[ u(t) = \begin{cases} 
1 & t > 0 \\
0 & t < 0 \\
\beta & t = 0 
\end{cases} \]

where \( 0 \leq \beta \leq 1 \).

The Laplace transform of \( u(t) \)

\[ F(s) = \int_{0^-}^{\infty} u(t) e^{-st} dt \]

\[ = \int_{0^-}^{0^+} \beta e^{-st} dt + \int_{0^+}^{\infty} (1) e^{-st} dt \]

\[ = \frac{\beta}{-s} e^{-st} \bigg|_{0^-}^{0^+} + \frac{1}{-s} e^{-st} \bigg|_{0^+}^{\infty} \]

\[ = -\frac{\beta}{s}(1 - 1) - \frac{1}{s}(e^{-\infty s} - e^{0.s}) \]

\[ = -\frac{1}{s}(e^{-\infty(\sigma + j\omega)} - 1) \]

\[ = -\frac{1}{s}(e^{-\infty\sigma} e^{-j\omega} - 1) \]

which converges to \( \frac{1}{s} \) if \( \sigma > 0 \) (since \( e^{-\infty\sigma} \to 0 \) and \( e^{-j\omega} \) is a finite complex number, as per Euler’s formula).

Therefore,

\[ u(t) \leftrightarrow \frac{1}{s} \quad (\text{Re}(s) > 0) \]
**Example.** Find the Laplace transform of the unit-impulse function, $\delta(t)$

$$f(t) = \delta(t) \rightarrow F(s) = ?$$

$$F(s) = \int_{0^-}^{\infty} \delta(t)e^{-st} dt = \int_{0^-}^{\infty} \delta(t)e^0 dt = \int_{0^-}^{\infty} \delta(t)dt = 1$$

Where the sifting property was used. Therefore, we have $L\{\delta(t)\} = 1$.

**Two Properties of Laplace Transform**

Given: $f(t) \leftrightarrow F(s)$

Then,

1. **Scaling Property:** $af(t) \leftrightarrow aF(s)$
2. **Addition Property:** $f_1(t) + f_2(t) \leftrightarrow F_1(s) + F_2(s)$

**Example.** Find the Laplace transform of $f(t) = e^{\lambda t}u(t)$. Assume complex $\lambda$.

$$f(t) = e^{\lambda t}u(t) \rightarrow F(s) = L\{e^{\lambda t}u(t)\} = ?$$

$$F(s) = \int_{0^-}^{\infty} e^{\lambda t}u(t)e^{-st} dt$$

$$= \int_{0^+}^{\infty} e^{\lambda t}(1)e^{-st} dt \quad \text{(since } f(t) \text{ has no impulsive component)}$$

$$\int_{0^+}^{\infty} e^{(\lambda-s)t} dt = \frac{1}{\lambda - s} e^{(\lambda-s)t} \bigg|_{0^+}^{\infty}$$

$$= \frac{1}{\lambda - s} (e^{(\lambda-s)\infty} - e^{0^+})$$

Therefore, $e^{\lambda t}u(t)$ transforms to $\frac{1}{s-\lambda}$ if $\text{Re}(\lambda - s) < 0$, or $\text{Re}(s) > \text{Re}(\lambda)$. 
A Short Table of (Unilateral) Laplace Transforms

<table>
<thead>
<tr>
<th>$f(t)$</th>
<th>$F(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 \delta(t)$</td>
<td>$1$</td>
</tr>
<tr>
<td>$2 u(t)$</td>
<td>(\frac{1}{s})</td>
</tr>
<tr>
<td>$3 tu(t)$</td>
<td>(\frac{1}{s^2})</td>
</tr>
<tr>
<td>$4 t^n u(t)$</td>
<td>(\frac{n!}{s^{n+1}})</td>
</tr>
<tr>
<td>$5 e^{\lambda t} u(t)$</td>
<td>(\frac{1}{s - \lambda})</td>
</tr>
<tr>
<td>$6 te^{\lambda t} u(t)$</td>
<td>(\frac{1}{(s - \lambda)^2})</td>
</tr>
<tr>
<td>$7 t^n e^{\lambda t} u(t)$</td>
<td>(\frac{n!}{(s - \lambda)^{n+1}})</td>
</tr>
<tr>
<td>$8a \cos bt u(t)$</td>
<td>(\frac{s}{s^2 + b^2})</td>
</tr>
<tr>
<td>$8b \sin bt u(t)$</td>
<td>(\frac{b}{s^2 + b^2})</td>
</tr>
<tr>
<td>$9a e^{-at}\cos bt u(t)$</td>
<td>(\frac{s + a}{(s + a)^2 + b^2})</td>
</tr>
<tr>
<td>$9b e^{-at}\sin bt u(t)$</td>
<td>(\frac{b}{(s + a)^2 + b^2})</td>
</tr>
<tr>
<td>$10a re^{-at}\cos (bt + \theta) u(t)$</td>
<td>(\frac{(r \cos \theta)s + (ar \cos \theta - br \sin \theta)}{s^2 + 2as + (a^2 + b^2)})</td>
</tr>
</tbody>
</table>

Note: The unilateral Laplace transform assumes that $f(t)$ is causal; i.e., $f(t) = f(t)u(t)$. 
Example. Show that:

\[ f(t) = \cos(\omega_0 t) u(t) \quad \leftrightarrow \quad \frac{s}{s^2 + \omega_0^2} \]

\[ F(s) = \int_{0^-}^{\infty} \cos(\omega_0 t) u(t)e^{-st} dt \]

\[ = \int_{0^+}^{\infty} e^{-st} \cos(\omega_0 t) dt \]

Your turn: Find the transform by solving the integral.

Alternatively, we may use Euler’s Formula to express the sinusoid in terms of exponentials, as follow:

\[ \cos(\omega_0 t) u(t) = \frac{1}{2} [e^{j\omega_0 t} + e^{-j\omega_0 t}] u(t) \]

\[ = \frac{1}{2} e^{j\omega_0 t} u(t) + \frac{1}{2} e^{-j\omega_0 t} u(t) \]

Recall that we have just derived \( e^{\lambda t} u(t) \leftrightarrow \frac{1}{s-\lambda} \). Setting \( \lambda = j\omega_0 \) leads to

\[ F(s) = \frac{1}{2} \cdot \frac{1}{s - j\omega_0} + \frac{1}{2} \cdot \frac{1}{s - (-j\omega_0)} \]

\[ = \frac{1}{2} \cdot \frac{1}{s - j\omega_0} + \frac{1}{2} \cdot \frac{1}{s + j\omega_0} \]

\[ = \frac{\left(\frac{1}{2}\right)(s + j\omega_0) + \left(\frac{1}{2}\right)(s - j\omega_0)}{s^2 + \omega_0^2} \]

\[ = \frac{s}{s^2 + \omega_0^2} \]
Pierre-Simon, marquis de Laplace (1749–1827) was an influential French scholar whose work was important to the development of mathematics, statistics, physics and astronomy. He summarized and extended the work of his predecessors in his five-volume Mécanique Céleste (Celestial Mechanics) (1799–1825). This work translated the geometric study of classical mechanics to one based on calculus, opening up a broader range of problems. In statistics, the Bayesian interpretation of probability was developed mainly by Laplace.

Laplace formulated Laplace's equation, and pioneered the Laplace transform which appears in many branches of mathematical physics, a field that he took a leading role in forming. The Laplacian differential operator, widely used in mathematics, is also named after him. He restated and developed the nebular hypothesis of the origin of the Solar System and was one of the first scientists to postulate the existence of black holes and the notion of gravitational collapse.

Laplace is remembered as one of the greatest scientists of all time. Sometimes referred to as the French Newton or Newton of France, he has been described as possessing a phenomenal natural mathematical faculty superior to that of any of his contemporaries.