Motor Control

Consider the problem of controlling a dc motor in such a way that its shaft’s angle at time $t$ is determined by the value of the input signal $f(t)$. For example, if we set $f(t) = 8$, then we expect the output angle $y(t) = \theta(t)$ to quickly change and stabilize at $8^\circ$. Ideally, we would like the motor angle to be equal to the input signal [i.e., $y(t) = f(t)$] for all time. In practice, we would expect $y(t)$ to *quickly and smoothly track* $f(t)$.

Of course, if we simply apply a constant voltage to the dc motor, the motor will keep spinning, implying that the system does not stabilize. Control theory explores the addition of negative feedback as a means of controlling the output of a given system, as is shown next.
The first step is to derive a differential equation that describes the (open loop) system. The following is a circuit model for a dc motor.

\[ \text{KVL: } -f(t) + Ri(t) + v_{emf}(t) = 0 \]  \hspace{1cm} (1)

Faraday’s Law: \( v_{emf}(t) = c_1 \frac{d\theta(t)}{dt} \)

Ampere’s Law: \( i(t) = c_2 J \frac{d^2\theta(t)}{dt^2} \)

where \( c_1, c_2 \) and \( J > 0 \) \hspace{1cm} (\( J \) is the load’s moment of inertia)

Substituting in equation (1), we arrive at the dynamical system

\[ \ddot{\theta}(t) + a\dot{\theta}(t) = bf(t) \]  \hspace{1cm} (2)

where \( a \equiv \frac{c_1}{c_2 RJ} > 0, \ b \equiv \frac{1}{c_2 RJ} > 0. \)

The transfer function of this system is (we will rename \( \theta \) as \( y \))

\[ H(s) = \frac{\theta(s)}{F(s)} = \frac{Y(s)}{F(s)} \]

To find \( H(s) \), we set \( y(0^-) = 0 \) and \( \dot{y}(0^-) = 0 \) and Laplace transform equation (2) to obtain

\[ s^2Y(s) + asY(s) = bF(s) \]
\[(s^2 + as)Y(s) = bF(s)\]

\[H(s) = \frac{Y(s)}{F(s)} = \frac{b}{s(s + a)}\]

In the remainder of this analysis, we will assume \(a = 8, b = 1\) which leads to:

\[H(s) = \frac{1}{s(s + 8)}\]

The poles of the motor are at \(s_1 = -8, s_2 = 0\), (i.e., natural modes are \(A_1 e^{-8t}\) and \(A_2 e^{0t} = A_2\)) thus rendering the motor critically stable.

The response \(y(t)\) is determined as \(y(t) = L^{-1}\{H(s)F(s)\}\). So, for a constant causal input, \(f(t) = \theta_0 u(t)\), the response is

\[y(t) = L^{-1}\left\{\frac{1}{s(s + 8)} \frac{\theta_0}{s}\right\} = \theta_0 L^{-1}\left\{-\frac{1}{64} + \frac{1}{8} + \frac{1}{64}e^{-8t}\right\}\]

which diverges for large \(t\), \(\lim_{t \to \infty} y(t) = \infty\). (Can you explain the \(\frac{t}{8}\) term?)

We want \(y(t)\) to track \(f(t)\); i.e. \(\lim_{t \to \infty} y(t) = \theta_0\). So, we employ negative feedback with amplification \((K > 0)\) as shown below.
Convergence occurs when $f(t) - y(t) = 0$. So, this strategy changes the signal that drives the motor to the error signal $f(t) - y(t)$. Thus, when $y(t)$ tracks $f(t)$, the input driving the motor is 0 Volt and the motor should stop spinning (what is the difference between zero input voltage and open input terminals on the motor?). However, if $y(t) < f(t)$, then the error signal is positive and is going to spin the motor in the direction that increases $y(t)$. On the other hand, if $y(t) > f(t)$ the error signal is negative and it will attempt to spin the motor in the opposite direction [the direction that decreases $y(t)$]. So, in both cases, the magnitude of the error signal driving the motor is reduced. The gain $K$ amplifies the error signal. As this goes on, one should keep in mind the loaded motor’s inertia that “resists” the motor’s acceleration/deceleration.

The stability of the above (closed loop) system is determined by the poles of its transfer function. The following is a derivation of the transfer function for the closed loop system (refer to the previous figure),

$$G(s) = \frac{Y(s)}{F(s)} = \frac{KH(s)[F(s) - Y(s)]}{F(s)}$$

$$G(s) = KH(s) - \frac{Y(s)}{F(s)} KH(s)$$

Noting that $G(s) = \frac{Y(s)}{F(s)}$, the above equation becomes

$$G(s) = KH(s) - G(s)KH(s), \text{ or}$$

$$[1 + KH(s)]G(s) = KH(s)$$

Thus, leading to the closed-loop transfer function,

$$G(s) = \frac{KH(s)}{1 + KH(s)}$$

Substituting $H(s) = \frac{1}{s(s+8)}$ in the above expression, gives
\[ G(s) = \frac{K}{s^2 + 8s + K} \]

which has its poles at \( s_{1,2} = -4 \pm \sqrt{16 - K} \).

The gain \( K \) plays a critical role in determining the stability and the tracking speed of the closed-loop system, as is illustrated next.

Case (i): \( 16 - K < 0 \) (i.e., \( K > 16 \)) \( \rightarrow s_{1,2} = -4 \pm j\sqrt{K - 16} \)

This leads to an *underdamped* step-response (the step response is a commonly accepted test signal for closed loop systems):

\[ y(t) = [1 + A_1 e^{-4t} \cos(\sqrt{K - 16} t) + A_2 e^{-4t} \sin(\sqrt{K - 16} t)]u(t) \]

\[ y(t) = [1 + A e^{-4t} \cos(\sqrt{K - 16} t + \theta)]u(t) \]

Case (ii): \( 16 - K = 0 \) (i.e., \( K = 16 \)) \( \rightarrow s_{1,2} = -4 \)

which leads to the *critically-damped* step-response:

\[ y(t) = [1 + A_1 e^{-4t} + A_2 te^{-4t}]u(t) \]

Case (iii): \( 16 - K > 0 \) (i.e., \( 0 < K < 16 \)) \( \rightarrow s_{1,2} \) are real and are located in the LHP. This leads to the *overdamped*, step-response:

\[ y(t) = \left[ 1 + A_1 e^{(-4+\sqrt{16-K})t} + A_2 e^{(-4-\sqrt{16-K})t} \right]u(t) \]

Case (iv): \( K < 0 \) \( \rightarrow s_1 < -4 \) and \( s_2 > 0 \) leading to an unstable response.

The following are examples of pole locations for various gain values.
The Root Locus

The root locus is a plot (similar to the above plots) that displays the trajectory (locus) of the poles of the feedback system as a function of the gain parameter, $K$. The root locus for the motor control problem is shown below (generated using Mathcad).

Alternatively, Matlab’s Control Toolbox can be used to generate the root locus plot. The command for that is `rlocus(num,den)`, where `num` is the
coefficient vector of the numerator polynomial, \( N(s) \), of the open-loop transfer function, \( H(s) = \frac{N(s)}{D(s)} \), and \( \text{den} \) is the polynomial coefficient vector of the denominator polynomial \( D(s) \). The generated root locus is that for the poles of the closed loop system, given by

\[
G(s) = \frac{KH(s)}{1 + KH(s)}
\]

Therefore, if we are interested in the root locus for the preceding closed loop motor control system, whose open-loop transfer function is

\[
H(s) = \frac{1}{s(s + 8)} = \frac{1}{s^2 + 8s + 0}
\]

we would just execute the instruction `rlocus([1],[1 8 0])`. Matlab then generates the following plot.

Your turn: Generate the root locus plot for the feedback system with gain \( K \), where the open-loop transfer function.

\[
H(s) = \frac{s^2 + 2s + 4}{s(s + 4)(s + 6)(s^2 + 1.4s + 1)}
\]

Estimate the range of \( K \) for which the system is stable.
Assume that the input to the above system is the unit-step function, \( u(t) \). Use Mathcad to solve for the (zero-state) response of the closed-loop system for the gain values: 80, 16 and 1.

**Solution:**

\[
\begin{align*}
  f(t) & := \Phi(t) \\
  F(s) & := f(t) \text{ laplace, } t \rightarrow \frac{1}{s}
\end{align*}
\]

**Transfer function of Motor and load:**

\[
H(s) := \frac{1}{s(s + 8)}
\]

**System transfer function with Unity feedback and pregain, \( K \):**

\[
G(s,K) := \frac{K \cdot H(s)}{1 + K \cdot H(s)} \text{ simplify } \rightarrow \frac{K}{(s^2 + 8s + K)}
\]

\[
K := 80 \\
\]

\[
s^2 + 8s + K \text{ solve, } s \rightarrow \begin{pmatrix} -4 + 8i \\ -4 - 8i \end{pmatrix}
\]

**System’s response (output motor angle), \( y(t) \):**

\[
y(t) := F(s) \cdot G(s,K) \text{ invlaplace, } s \rightarrow 1 - \exp(-4t) \cdot \cos(8t) - \frac{1}{2} \cdot \exp(-4t) \cdot \sin(8t)
\]
Your turn: Solve for the above result analytically; i.e., show that:

\[ y_{zs}(t) = L^{-1}\left\{ \frac{80}{s(s^2 + 8s + 80)} \right\} \]

\[ K := 16 \]

\[ s^2 + 8s + K \text{ solve, } s \rightarrow \begin{pmatrix} -4 \\ -4 \end{pmatrix} \]

System's response (output motor angle), \( y(t) \):

\[ y(t) := F(s) \cdot G(s, K) \text{ invlaplace, } s \rightarrow 1 - 4 \cdot t \cdot \exp(-4 \cdot t) - \exp(-4 \cdot t) \]
Your turn: Show that the following two expressions are equivalent,

\[ 1 - \exp(-4t) \cdot \cosh(\sqrt{15}t) - \frac{4}{15} \cdot \exp(-4t) \cdot \sqrt{15} \cdot \sinh(\sqrt{15}t) \]

\[ 1 + \left( 2 \frac{\sqrt{15}}{15} - \frac{1}{2} \right) e^{-(\sqrt{15}+4)t} - \left( 2 \frac{\sqrt{15}}{15} + \frac{1}{2} \right) e^{(\sqrt{15}-4)t} \]

Hint: \( \cosh(x) = \frac{e^x + e^{-x}}{2} \), \( \sinh(x) = \frac{e^x - e^{-x}}{2} \)
The following is the closed-loop system response to a piece-wise linear input signal for two different gain values ($K = 7$ and 80).

Your turn: Employ Mathcad simulations to reproduce the above plots.
Matlab Simulations

**Impulse response, step response, and ramp response**

```
num=[80]; den=[1 8 80]; impulse(num, den)
```

![Impulse Response](image1)

```
num=[80]; den=[1 8 80]; step(num, den)
```

![Step Response](image2)
Your turn: Consider the following feedback system.

Show that the overall system transfer function (with feedback) \( H_{FB}(s) \) is given by

\[
H_{FB}(s) = \frac{Y(s)}{F(s)} = \frac{H(s)}{1 - H(s)G(s)}
\]

Your turn: Employ the above result to determine the overall transfer function 
\( H(s) = \frac{Y(s)}{F(s)} \) for the following systems.
System Interconnection

Consider two LTI systems whose unit-impulse responses are $h_1(t)$ and $h_2(t)$ [or, equivalently, transfer functions $H_1(s)$ and $H_2(s)$]. If we cascade the two systems (does not matter which one comes first) then the overall unit-impulse response would be the convolution $h(t) = h_1(t)\ast h_2(t)$; and the overall transfer function would be the product $H(s) = H_1(s)H_2(s)$ (as long as the impedance of the second system is very high, so it would not load the first system). This is illustrated in the following figure.

Similarly, for two systems connected in parallel, the overall unit-impulse response is given by the sum, $h(t) = h_1(t) + h_2(t)$, and the overall transfer function is $H(s) = H_1(s) + H_2(s)$.
Your turn: Show that the above four system equivalencies are valid.

Here is an equivalence proof for the first system:

Let \( g(t) = f(t) \ast h_1(t) \), then the output of the cascade of the two systems is

\[
y(t) = g(t) \ast h_2(t) = [f(t) \ast h_1(t)] \ast h_2(t) = f(t) \ast [h_1(t) \ast h_2(t)]
\]

Therefore, the overall system can be thought of one having the unit-impulse response \( h(t) = h_1(t) \ast h_2(t) \).
Physical System Realization of dc Motor with Controller

The following figure shows a picture of a dc motor control trainer (typically used in undergraduate level control laboratories. This particular system is made by Feedback Instruments, UK). The circuit board shown implements a PID controller (refer to the appendix for details).

Watch the movie of the controller in action:
Realization of Transfer Functions Using Electric Circuits

Differentiator Circuit

Let us say we are interested in realizing the transfer function \( H(s) = s \), or

\[ V_0(s) = sV_{in}(s) \]

Taking the inverse Laplace transform of this equation leads to

\[ v_0(t) = \frac{dv_{in}(t)}{dt} \]

The following is an operational amplifier (op-amp) circuit that differentiates its input:

KCL at 1: \( 0 + \frac{0 - V_0}{R} + \frac{0 - V_{in}}{\frac{1}{sc}} = 0 \)

\[ V_0(s) = -CRsV_{in}(s) \]

If we set \( CR = 1 \), we obtain,

\[ V_0(s) = -sV_{in}(s) \]

or,

\[ H(s) = \frac{V_0(s)}{V_{in}(s)} = -s \]
**Inverting Amplifier Circuit**

The following op-amp circuit amplifies its input voltage by the negative gain \(-\frac{R_f}{R}\),

\[
\begin{align*}
R_f & \quad v_{in}(t) \quad R \\
\text{--} & \quad \text{+} \\
\quad & \quad \text{+} \\
\text{+} & \quad v_{o}(t) = -\left(\frac{R_f}{R}\right)v_{in}(t)
\end{align*}
\]

If we set \(R_f = R\), we obtain an inverting circuit whose transfer function is

\[
H(s) = \frac{V_o(s)}{V_{in}(s)} = -1
\]

Now, the transfer function \(H(s) = s\) can be realized by a first stage differentiator followed by a second stage inverting amplifier. The resistor values used in the inverter circuit must have large resistance (on the order of 10KΩ or higher), so that the second stage does not load the first stage.
Alternatively, a buffer circuit can be used between two circuit stages to prevent loading effects. The following op-amp buffer circuit has the required high-input resistance. Its transfer function is $H(s) = 1$.

![Buffer Circuit Diagram]

**Integrator Circuit**

An op-amp circuit who’s output is given by,

$$v_o(t) = \frac{-1}{RC} \int_{0^-}^{t} v_{in}(\tau) d\tau - v_o(0^-)$$

is shown below.

**Your Turn:** Derive the equation from the Laplace circuit, where $v_o(0^-)$ is the initial voltage across the capacitor.
The transfer function for this circuit is (set \( v_c(0^-) = 0 \) and use the integration property of the Laplace transform),

\[
H(s) = \frac{V_o(s)}{V_{in}(s)} = -\frac{1}{RC} \frac{1}{s}
\]

and if \( RC = 1 \), the above expression becomes,

\[
H(s) = -\frac{1}{s}
\]

The **Summing Integrator** is the basis for an analog computer:

It has the following input/output relationship,

\[
v_o(t) = -\int_{0^-}^{t} \left[ \frac{1}{R_1 C} v_{in1}(\tau) + \frac{1}{R_2 C} v_{in2}(\tau) + \frac{1}{R_3 C} v_{in3}(\tau) \right] d\tau - v_o(0^-)
\]
**Differential Amplifier Circuit**

The following circuit amplifies the difference of its inputs, and would be useful in realizing the amplified difference portion of a negative feedback system.

![Differential Amplifier Circuit Diagram](image)

\[ v_0(t) = \frac{R_f}{R} (v_1 - v_2) \]

**Passive Realization of the Transfer Function of a dc Motor**

Passive circuits can be used to synthesize transfer functions. The following \( RL \) circuit

![Passive R-L Circuit Diagram](image)

has the transfer function (easily obtained by treating the circuit as a voltage divider)
If we set \( R = 8 \Omega \) and \( L = 1 \text{H} \), we obtain

\[
H_1(s) = \frac{8}{s + 8}
\]

Now, we may cascade an op-amp-based integrator (whose transfer function is \( H_2(s) = -\frac{1}{8s} \)) to the right of the above passive circuit as shown below,

\[
H_1(s) \cdot H_2(s)
\]

to obtain the overall circuit transfer function

\[
H(s) = H_1(s)H_2(s) = \frac{-1}{s(s + 8)}
\]

This last result corresponds to the transfer function of the dc motor used in the control system (introduced at the beginning of this lecture). The only difference is the extra gain, \(-1\). An inverting amplifier with gain \(-1\) may be used in cascade in the above system to change the overall transfer function gain to +1.

Refer to the last part of this lecture for a synthesis method of a passive circuit having a transfer function of the form \( H(s) = \frac{1}{D(s)} \).
Realization of the dc Motor’s Closed Loop System

Employing the component circuits introduced above, we may formulate a block diagram for the complete feedback control system as shown below.

The following is a circuit simulation of the above system ($K = 16$).
Simulation with gain $K = 7$. 
Simulation with $K = 80$,
The following figure depicts the closed-loop system’s response to a unit-step input, for different gain values. (Underdamped; critically damped; overdamped)

Example. A parallel $RLC$ circuit with overdamped response. The circuit employs a relay to automatically trigger the charge/discharge cycle (thus the oscillatory behavior). https://www.youtube.com/watch?v=WuG4nOyF99s

The circuit parameters are: Lamp ($1.5\Omega$), $C = 2200\mu F$ and Coil ($L = 180\text{mH}$ and $R = 300\Omega$).

Your turn: Verify analytically that the discharge cycle is overdamped.
Matlab/Simulink Solution
(Matlab’s student comes with Simulink. Here is a tutorial on how to use Simulink: http://www.mathworks.com/help/simulink/)
$K = 16$

$K = 80$
It should be noted that the gain constant $K$ has physical constraints that impose an upper limit on its value. For example, if the voltage gain $K$ is realized using an operational amplifier circuit then the output of the amplifier will saturate at $\pm V_{Sat}$ ($V_{Sat}$ is close to the operational amplifier supply voltage). So, a proper simulation must include a nonlinear voltage limiter (saturation block) placed just after the gain block. Once nonlinearities are used in a control system model, we lose the usefulness of Laplace transform analysis, and the response of the system must be determined employing numerical methods, or simulators such as Simulink.

It should also be noted here that the typical operational amplifier can’t drive the motor directly, because a dc motor requires relatively high currents (especially, when it is stalled). So, in practice, a power amplifier (of unity voltage gain) must be used between the gain amplifier and the motor.

The following simulation compares the theoretical closed-loop response (red trace; $K = 80$) of the closed-loop dc motor system to that with a clamped amplifier output voltage (blue trace; $K = 80, V_{Sat} = \pm 12$).
Example. Commercial aircraft pitch angle. In Lecture 11 we had determined that the following dynamical system is unstable,

\[ H(s) = \frac{\Theta(s)}{F(s)} = \frac{1.151s + 0.1774}{s(s^2 + 0.739s + 0.921)} \]

where, \( \theta(t) = L^{-1}\{\Theta(s)\} \) is the pitch angle and \( f(t) = L^{-1}\{F(s)\} \) is the elevator deflection angle. The objective is to control the aircraft pitch angle and make it track \( f(t) \).

Let us apply simple negative feedback where the input to the system is made equal to the difference signal, \( f(t) - \theta(t) \). The resulting closed-loop system (with unity gain) has the transfer function

\[ G(s) = \frac{H(s)}{1 + H(s)} = \frac{1.151s + 0.1774}{s^3 + 0.739s^2 + 2.072s + 0.1774} \]

The poles of the closed loop system are the solution to the polynomial equation

\[ s^3 + 0.739s^2 + 2.072s + 0.1774 = 0 \]

This system is stable since all poles are in the LHP.

The zero-state response for the input \( f(t) = 3u(t) \) can be solved for, using Mathcad, to obtain
which can be expressed as

\[ \theta_{zs}(t) = [3 - 1.319e^{-0.088t} + 1.748e^{-0.841t} \cos(1.382t + 2.863)]u(t) \]

The closed loop, pitch angle zero-state response is shown in the following plot. After as brief oscillatory period, the steady-state angle, \( \theta_{ss}(t) \), asymptotically approaches the 3° elevator deflection angle.

![Graph](image)

**Your turn:** Repeat the above aircraft pitch angle analysis employing a gain \( K \), which results in the closed-loop transfer function

\[
G(s) = \frac{KH(s)}{1 + KH(s)}
\]

Employ Simulink and perform simulations with \( K = 0.5, 1 \) and 1.5

In practice, the gain block, \( K \), is not appropriate for controlling the system at hand and it must be replaced with a controller having a rational transfer function, \( H_{PID}(s) \). A relatively simple type of controller that is common in the industry is known as PID controller. Refer to the appendix for an introduction and an example.
Active Analog (op-amp) Circuit Realization of Transfer Functions

Next, we design an active circuit realization of the linear system,

\[ H(s) = \frac{Y(s)}{F(s)} = \frac{1}{s + 8} \]

The above transfer function can be expressed as

\[ sY(s) + 8Y(s) = F(s) \]

Taking the inverse Laplace transform (recall that the initial conditions are assumed to be zero here) we get,

\[ \dot{y}(t) + 8y(t) = f(t) \]

or

\[ \dot{y}(t) = f(t) - 8y(t) \]

Integrating the above equation leads to,

\[ y(t) = \int_{0}^{t} (f(\tau) - 8y(\tau))d\tau \]

which can be realized as follows [this feedback formulation technique is due to Lord Kelvin (1875)],

We could have also arrived at the above block diagram by working in the frequency domain:

\[ sY(s) + 8Y(s) = F(s) \Rightarrow sY(s) = F(s) - 8Y(s) \]

or,

\[ Y(s) = \frac{1}{s} [F(s) - 8Y(s)] \]
Had we started with the formulation

\[ y(t) = \frac{1}{8} [f(t) - \dot{y}(t)] \]

the realization would have been given as shown below (note differentiator)

However, the integrator-based implementation of the transfer function is much more preferable over the differentiator-based one, because of the problems arising from differentiating noisy signals.

**Your turn:** Give an op-amp realization for the system \( H(s) = \frac{2s+5}{s^2+4s+10} \) by (directly) realizing the corresponding differential equation,

\[ \ddot{y}(t) + 4\dot{y}(t) + 10y(t) = 2\dot{f}(t) + 5f(t) \]

Hint: First, solve for \( y(t) \),

\[ y(t) = \int \left[ -4y(\tau) + 2f(\tau) + \int (-10y(\tau) + 5f(\tau))d\tau \right] d\tau \]
Canonical System Realization

Another way of realizing a transfer function \( H(s) = \frac{N(s)}{D(s)} \) is to treat it as a cascade of two systems

\[
H(s) = \frac{Y(s)}{F(s)} = H_1(s)H_2(s) = \frac{1}{D(s)}N(s)
\]

We will write \( H_1(s) = \frac{X(s)}{F(s)} = \frac{1}{D(s)} \) and \( H_2(s) = \frac{Y(s)}{X(s)} = N(s) \).

Next, we illustrate this realization approach for \( H(s) = \frac{2s+5}{s^2+4s+10} \).

Here, \( H_1(s) = \frac{X(s)}{F(s)} = \frac{1}{s^2+4s+10} \) and \( H_2(s) = \frac{Y(s)}{X(s)} = 2s + 5 \)

and we may express the input/output relation for each subsystem as, respectively,

\[
(s^2 + 4s + 10)X(s) = F(s) \quad \text{and} \quad Y(s) = (2s + 5)X(s)
\]

The following block diagram represents the first subsystem in terms of integrator blocks, \( \frac{1}{s} \)

\[
s^2X(s) = F(s) - 4sX(s) - 10X(s)
\]
The second subsystem generates $Y(s)$ from $X(s)$, according to the equation

$$Y(s) = 2sX(s) + 5X(s)$$

Therefore, the overall system’s block diagram is given by

This diagram may now be converted into a diagram involving integrators, inverting amplifiers and inverting summers (keep in mind that an op-amp integrator also inverts its input)
The complete op-amp circuit (canonical) realization is shown below employing practical component values.

![Op-amp circuit](image)

**Your turn:** Design a canonical op-amp realization for the system

\[
H(s) = \frac{s^2 + 5s + 2}{s^2 + 4s + 13}
\]

**Parallel System Realization**

The canonical system realization tends to be sensitive to small parameter changes in the system. The reason behind this undesirable sensitivity is that all coefficients interact with each other, and a change in any component (say resistor value) will be magnified through its repeated influence from feedback and feedforward connections.

Alternatively, a less sensitive realization is based on a parallel architecture. For example, we may implement the transfer function

\[
H(s) = \frac{4s+28}{s^2+6s+5} \quad \text{(after performing partial fraction expansion)}
\]
\[ H(s) = \frac{4s + 28}{s^2 + 6s + 5} = \frac{6}{s + 1} - \frac{2}{s + 5} \]

in parallel form, as

\[
\begin{array}{c}
\text{F}(s) \rightarrow \frac{6}{s + 1} \rightarrow \frac{2}{s + 5} \rightarrow \sum \rightarrow Y(s)
\end{array}
\]

**Your turn:** Generate a block diagram realization (employing integrator, gain and summing blocks) for the above system.

**Your turn:** Use Simulink to generate (on a virtual scope) the zero-state response of the system \( H(s) = \frac{4s+28}{s^2+6s+5} \) to the input \( f(t) = u(t) - u(t-2) \). Verify your result by solving for the response (analytically) employing the Laplace transform and then plot it.

Simulink Answer:
The direct (differential equation based), canonical and parallel realizations by no means lead to the use of the smallest number of op-amps. There are circuits (such as the Sallen-Key, see below) that are capable of realizing a second-order transfer function using only one op-amp.

\[
H(s) = \frac{1}{s^2 + \left(\frac{1}{R_1 C_1} + \frac{1}{R_2 C_1}\right)s + \frac{1}{R_1 R_2 C_1 C_2}}
\]

Refer to Lecture 19 for the design of such circuits.
Passive Circuit Synthesis of the Transfer Function $H(s) = \frac{1}{D(s)}$

Consider the transfer function $H(s) = \frac{1}{D(s)}$ with $D(s)$ a Hurwitz polynomial. Any polynomial $D(s)$ can be expressed as the sum $p(s) + q(s)$ where $p(s)$ consists of all even powers of $s$ and $q(s)$ consists of all the odd powers of $s$. For example, if $D(s) = s^3 + 2s^2 + 2s + 1$, then $p(s) = 2s^2 + 1$ and $q(s) = s^3 + 2s$. If the roots of $p(s)$ and $q(s)$ are pure imaginary and they alternate then $D(s)$ is a Hurwitz polynomial.

For example, the $D(s) = s^3 + 2s^2 + 2s + 1$ is a Hurwitz polynomial because: (1) The roots of $p(s) = 2s^2 + 1$ are (pure imaginary): $\pm j\sqrt{1/2}$. (2) the roots of $q(s) = s^3 + 2s$ are (pure imaginary) $0, \pm j\sqrt{2}$. (3) the roots of $p(s)$ and $q(s)$ alternate:

$$-\sqrt{2} < -\sqrt{\frac{1}{2}} < 0 < \sqrt{\frac{1}{2}} < \sqrt{2}.$$ 

**Your turn:** Show that $D(s) = s^4 + 2s^3 + 10s^2 + 8s + 9$ is a Hurwitz polynomial, but $D(s) = s^3 + s^2 + 9s + 25$ is not.

It can be shown that when $D(s) = p(s) + q(s)$ is Hurwitz, then the transfer function $H(s) = \frac{1}{D(s)}$ can be realized with a passive $LC$ ladder network terminated by a $1\Omega$ resistor. In this case, the output impedance $Z(s)$ (the impedance of the circuit looking into the unterminated circuit output) can be shown to be $Z(s) = \frac{p(s)}{q(s)}$. The synthesis method is illustrated with the following example.
Consider the transfer function $H(s) = \frac{1}{s^3 + 2s^2 + 2s + 1}$. We showed above that $D(s)$ is Hurwitz. Now, we form the ratio $Z_a(s) = \frac{p(s)}{q(s)} = \frac{s^3 + 2s}{2s^2 + 1}$ and express it as a continued fraction in the form,

$$Z_a(s) = k_0s + \frac{1}{l_0 + \frac{1}{k_1 + \frac{1}{l_1 + \frac{1}{k_2 + \ddots}}}}$$

Applying continued fraction to our impedance (divide, flip remainder and divide, flip remainder and divide, etc.), we obtain:

$$Z_a(s) = \frac{p(s)}{q(s)} = \frac{s^3 + 2s}{2s^2 + 1} = \frac{1}{2}s + \frac{\frac{3}{2}s}{2s^2 + 1} = \frac{1}{2}s + \frac{1}{\frac{3}{2}s + \frac{3}{2}s}$$

Now we can view this impedance (looking into the output terminals) as a series inductor $L_1 = \frac{1}{2}$ in series with the parallel combination of a $C = \frac{4}{3}$ capacitor and an $L_2 = \frac{3}{2}$ inductor. As shown in the figure below.
Since the last component (on the left) in the circuit is an inductor, the rule is to break the connection and insert a voltage source. Finally, the circuit is completed by applying a 1Ω load resistor to the output terminal. The circuits output is then the voltage across the resistor. The complete circuit is shown below.

We can verify our circuit by applying mesh analysis to the s-domain circuit (setting all initial energy to zero) to obtain,

\[
\left(\frac{3s}{2} + \frac{3}{4s}\right)I_1(s) - \frac{3}{4s}I_2(s) = E(s) \\
-\frac{3}{4s}I_1(s) + \left(1 + \frac{s}{2} + \frac{3}{4s}\right)I_2(s) = 0
\]

Solving for \(H(s) = \frac{V_R(s)}{E(s)} = \frac{RI_2(s)}{E(s)}\) results in the desired transfer function,

\[
H(s) = \frac{1}{s^3 + 2s^2 + 2s + 1}
\]
Alternatively, we could have started with the reciprocal impedance $Z_a(s)$,

$$Z_a(s) = \frac{p(s)}{q(s)} = \frac{2s^2 + 1}{s^3 + 2s} = \frac{1}{\frac{s}{2} + \frac{1}{\frac{4s + \frac{1}{2}}{s}^s}} = \frac{1}{\frac{1}{2s} + \frac{1}{\frac{4s + \frac{1}{2}}{s}^s}}$$

Now we can view this impedance (looking into the output terminals) as a shunt capacitor $C_1 = \frac{1}{2}$ in parallel with the series combination of a $C_1 = \frac{3}{2}$ capacitor and an $L = \frac{4}{3}$ inductor. As shown in the figure below.

Since the last component (on the left) in the above circuit is a capacitor, the rule is to apply the input as a current source in parallel with that capacitor. Finally, the circuit is completed by applying a 1Ω load resistor to the output terminal. The output signal is the voltage across the resistor. The complete circuit is shown below.
We can verify our resulting circuit by applying nodal analysis to the $s$-domain circuit (setting all initial energy to zero) to obtain,

$$
\left(\frac{3s}{2} + \frac{3}{4s}\right)V_1(s) - \frac{3}{4s}V_2(s) = I_g(s)
$$

$$
-\frac{3}{4s}V_1(s) + \left(1 + \frac{s}{2} + \frac{3}{4s}\right)V_2(s) = 0
$$

Solving for $H(s) = \frac{V_2(s)}{I_g(s)}$ results in the desired transfer function,

$$
H(s) = \frac{1}{s^3 + 2s^2 + 2s + 1}
$$

Note: If any of the coefficients in the continued fraction expansion turns out to be negative, then we will not be able to realize the function using passive components. Fortunately, a Hurwitz $D(s)$ guarantees that all coefficients are positive.

One advantage of passive circuit realization of linear systems (filters) is that they work well for high-frequency applications (> 1MHz). However, drawbacks include their inability to generate energy, so the power gain of a passive filter cannot exceed 1. Active filters (with op-amps) can be designed to provide significant gain. They also do not require inductors. One problem with using inductors is their relatively large three-dimensional footprint. Whereas capacitors and resistors and op-amps can be easily fabricated in planar form on machine-assembled printed circuit boards, inductors are generally more expensive to fabricate and more difficult to integrate into the rest of the circuit. On the other hand, op-amps
generally do not perform reliably at frequencies above 1MHz (the assumption of infinite gain no longer holds), so their use is limited to lower frequencies. Fortunately, inductor sizes become less of a problem above 1MHz since there impedance \( Z_L = j\omega L \) would allow a smaller value for \( L \), leading to a physically smaller inductor. So, passive \( LC \) filters are the predominant type used at higher frequencies. We will learn more about filters and filter design in Lecture 19.

Your turn: Show that the transfer function

\[
H(s) = \frac{1}{s^4 + 2s^3 + 10s^2 + 8s + 9}
\]

has the two equivalent realizations:
Appendix: PID Controller

A proportional–integral–derivative controller (PID controller) is a control block with transfer function $H_{PID}(s)$, commonly used in industrial closed-loop control systems. A PID controller continuously calculates an error value as the difference between a desired signal $f(t)$ and a measured process variable [usually system output $y(t)$] and applies a correction based on proportional, integral and derivative terms (denoted $P$, $I$, and $D$, respectively). The transfer function of a PID control block is

$$H_{PID}(s) = P + I \frac{1}{s} + Ds$$

where $P$, $I$ and $D$ (all non-negative) denote the coefficients for the proportional, integral and derivative terms, respectively. In this model:

- $P$ accounts for present values of the error. For example, if the error is large and positive, the control output will also be large and positive.
- $I$ accounts for past values of the error. For example, if the output is not sufficiently strong, the integral of the error will accumulate over time, and the controller will respond by applying a stronger action.
- $D$ accounts for possible future trends (prediction) of the error, based on its current rate of change.

In practice (and in the built-in Simulink “PID Controller” block), the derivative term is problematic (becomes very large) when the signal is noisy and/or has sharp variations. For this reason, the differentiator component is disabled for high frequencies by passing its output through a low-pass filter having transfer function $\frac{N}{s+N} = \frac{1}{1+ns}$, with high cutoff frequency ($N >> 1$). This allows us to express the PID controller as

$$H_{PID}(s) = P + I \frac{1}{s} + D \frac{s}{\frac{1}{N}s + 1}$$

A basic block diagram for the feedback controlled system utilizing a PID controller is shown below.
or equivalently,

The effects of varying the PID controller parameters are nicely captured in the following animation: [https://en.wikipedia.org/wiki/File:PID_Compensation_Animated.gif](https://en.wikipedia.org/wiki/File:PID_Compensation_Animated.gif)

The effects of independently increasing the PID controller parameters on the controlled system behavior are summarized in the following table.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Rise time</th>
<th>Overshoot</th>
<th>Settling time</th>
<th>Steady-state error</th>
<th>Stability</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P )</td>
<td>Decrease</td>
<td>Increase</td>
<td>Small change</td>
<td>Decrease</td>
<td>Degrade</td>
</tr>
<tr>
<td>( I )</td>
<td>Decrease</td>
<td>Increase</td>
<td>Increase</td>
<td>Eliminate</td>
<td>Degrade</td>
</tr>
<tr>
<td>( D )</td>
<td>Minor change</td>
<td>Decrease</td>
<td>Decrease</td>
<td>No effect in theory</td>
<td>Improve if ( D ) small</td>
</tr>
</tbody>
</table>

PID optimization software is available for computing/tuning the PID parameters. But, in many situations, an experienced control engineer/technician can manually tune these parameters. The idea is to set these parameters so that the system is stable, has minimal rise time, minimal overshoot, fast settling time and very small steady-state error. The following heuristic method (known as Ziegler-Nichols method) can be used.

Initially, the \( I \) and \( D \) gains are first set to zero. The proportional gain, \( P \), is increased (say, starting from zero) until it reaches the value, \( K_u \), at which the output of the closed loop system has stable and consistent oscillations. We will refer to the oscillation period as, \( T_u \). Then, the PID parameters are set according to the following (classic PID) formulas (typically, \( N \) is set to a large value, say 100):
\[ P = 0.6K_u, \quad I = \frac{1.2}{T_u} K_u, \quad D = \frac{3T_u}{40} K_u \]

There are other variations of the Zeigler-Nichols heuristic PID settings that are designed to meet certain desirable performance criteria. Such settings are summarized in the following table and apply to the PID controller:

\[
H_{PID}(s) = P + I \frac{1}{s} + D \frac{s}{\frac{1}{N} s + 1} = K_p \left( 1 + \frac{1}{T_i s} + \frac{T_d s}{\frac{1}{N} s + 1} \right) \\
= K_p \left( \frac{1}{N} + T_d \right) s^2 + \left( \frac{1}{NT_i} + 1 \right) s + \frac{1}{Ti}
\]

<table>
<thead>
<tr>
<th>Control Type</th>
<th>(K_p)</th>
<th>(T_i)</th>
<th>(T_d)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(P)</td>
<td>0.5(K_u)</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>(PI)</td>
<td>0.45(K_u)</td>
<td>(T_u/1.2)</td>
<td>-</td>
</tr>
<tr>
<td>(PD)</td>
<td>0.8(K_u)</td>
<td>-</td>
<td>(T_u/8)</td>
</tr>
<tr>
<td>classic PID</td>
<td>0.6(K_u)</td>
<td>(T_u/2)</td>
<td>(T_u/8)</td>
</tr>
<tr>
<td>Pessen Integral Rule</td>
<td>0.7(K_u)</td>
<td>(T_u/2.5)</td>
<td>(3T_u/20)</td>
</tr>
<tr>
<td>some overshoot</td>
<td>0.33(K_u)</td>
<td>(T_u/2)</td>
<td>(T_u/3)</td>
</tr>
<tr>
<td>no overshoot</td>
<td>0.2(K_u)</td>
<td>(T_u/2)</td>
<td>(T_u/3)</td>
</tr>
</tbody>
</table>

The following is a PID controller simulation for the zero-state response of the commercial aircraft system discussed earlier in this lecture [with \(f(t) = 3u(t)\)]. The PID parameters were determined using the classical Ziegler-Nichols method, with the aid of Simulink.
**Your turn:** Employ the (classic) Ziegler-Nichols method and Simulink to determine the PID coefficients and generate the above plot. Derive the overall closed-loop transfer function $G(s) = \frac{Y(s)}{F(s)}$ and determine its poles and zeros for the two PID controllers: $(P_1, I_1, D_1) = (0.6, 0.24, 0.375)$ and $(P_2, I_2, D_2) = (0.5, 0.12, 0)$. Assume $N = 100$. Employ Mathcad to solve for $y_{zs}(t)$ [the pitch angle], when subjected to the step input, $f(t) = u(t)$, for both PID controllers. Plot the two responses on the same graph. Your plot should look like this plot:

![Plot](image)

**Your turn:** Generate a root locus plot for the following closed loop system. Sweep $K$ from 0 to 200 with 0.5 increments. From the plot, estimate the range of gain values that lead to a stable system.

![Block Diagram](image)

$$H(s) = \frac{s + 1}{s(s - 1)(s^2 + 4s + 16)}$$

Recall that the root locus is a plot of the trajectory of the poles of the system,

$$G(s) = \frac{KH(s)}{1 + KH(s)}$$

Ans. $23.4 < K < 35.7$
**Your turn:** Show that the transfer function of the following operational amplifier based circuit is equivalent to that of the PID controller:

\[ H_{PID}(s) = P + I \frac{1}{s} + Ds \]

Express \( P, I \) and \( D \) in terms of the circuit element values.

**Ans.**

\[ P = \left( \frac{R_4}{R_3} \right) \left( \frac{R_1C_1 + R_2C_2}{R_1C_2} \right) \quad I = \left( \frac{R_4}{R_3} \right) \frac{1}{R_1C_2} \quad D = \left( \frac{R_4}{R_3} \right) R_2C_1 \]

**Your turn:** Show that the transfer function of the following circuit is equivalent to that of a PID controller:

\[ H_{PID}(s) = P + I \frac{1}{s} + D \frac{s}{\frac{1}{N}s + 1} \]

Express \( P, I, D \) and \( N \) in terms of the circuit element values.

**Ans.**

\[ P = \left( \frac{R_4}{R_3} \right) \frac{R_2(R_1 + R)}{R_1R} \quad I = \left( \frac{R_4}{R_3} \right) \frac{1}{R_1C_2} \quad D = \left( \frac{R_4}{R_3} \right) \frac{C_1R - C_2R_2}{C_2R} \quad N = \frac{1}{RC_1} \]
Propeller PID-Controlled Inverted Pendulum

Demonstration:

https://youtu.be/X2ELSFnOlUM
Mini Project

Consider the dc motor with transfer function

\[ H(s) = \frac{1}{s(s + 8)} \]

We have studied the simple negative feedback with error amplification of \( K \) earlier in this lecture. It was determined that the fastest non-oscillatory unit-step response corresponded to a gain \( K = 16 \) (critically damped response). We may view this controller as a PID controller with gain only (That is a proportional controller or P controller). Employ Simulink to answer the following:

a. Design a classic Ziegler-Nichols-based PID controller. Simulate the system (employing Simulink’s PID Controller block) and generate a plot of the unit-step response (assume \( N = 100 \)). Set the \( x \)-axis range to [0 5] and the \( y \)-axis range to [0 1.2]. The same plot should also include the step-response of the P-controlled system with gain \( P = 16 \), for comparison purposes.

b. Employ the tuning feature of Simulink to automatically tune your PID controller from Part a in order to achieve an improved PID controller. [You can access the tuning feature from within the PID controller block menu. After Simulink finishes tuning the PID controller, you may update the PID parameters by clicking on the “RESULTS” button, located at the upper right hand side corner of the PID controller menu, and then selecting “Update Block”]. Simulate the resulting PID controller and display its unit-step response on the same plot generated in Part a. Your plot should look similar to the following plot.

c. Modify the tuned PID controller obtained in Part b by setting the sliders inside the PID controller menu as shown below, and then display the resulting step-response on the same plot as before (allowing you to compare the step-responses of your four controllers).

d. Design the Operational Amplifier PID circuit (the bottom circuit, two slides back) so that it implements the PID controller form Part a. Assume \( R_1 = 100K\Omega \) and \( N = 200 \).

e. Employ Mathcad to determine the transfer function, its poles and the unit-impulse response for the closed loop system obtained in Part c.
The Segway people mover can be roughly modeled as a frictionless inverted pendulum on a cart with the following linear dynamics (assuming small angles):

\[
\theta''(t) - \frac{g}{L} \theta(t) = -\frac{1}{L} \theta'(t)
\]

Where \(\theta(t)\) is the angle (in radian) from the vertical, \(x(t)\) is the displacement of the cart (in meters), \(L\) is the length (in meters) of the pendulum and \(g\) is gravitational acceleration.

a. Derive the transfer function \(H(s)\) for the system and show that it is not stable.

b. Let \(K_1\) and \(K_2\) be two positive constants. And let \(K_1 + \frac{K_2}{s} = \frac{K_1 s + K_2}{s}\) be the transfer function of a controller that is used to stabilize the system (see Figure below). Derive the transfer function of the closed look system (i.e., the new \(Y(s)/X(s)\)).

c. For what values of \(K_1\) and \(K_2\) is the system stable?

d. Solve for the value of \(K_2\) (as a function of \(L, K_1\) and \(g\)) that leads to a critically damped response.

e. Solve (analytically) and plot (on the same graph) the response \(y(t)\) of the system for the input \(x(t) = 0.1 u(t)\) and \(K_2\) equal (i) 2, (ii) \(2\sqrt{g}\) and (iii) 10. Assume \(L = 1\), \(K_1 = 2\), and \(g = 9.8\). Also, assume that the system has zero initial conditions.

f. Verify your work using Mathcad and Simulink.
Mini Project

Use what you have learned so far in ECE 4330 to design, verify (by simulation using two or more of the following tools: Matlab, Mathcad, Simulink and Multisim) and build a linear electric circuit that displays and sustains a perfect circle on an oscilloscope. The circuit must not employ external signal generators, but it can employ a dc power supply (say 9Volt batteries). Bring your circuit to your instructor’s office for evaluation (an oscilloscope will be made available). Also, provide a well-written report.

Practical considerations when building an OpAmp-based integrator:

In a practical integrator circuit, a large resistance $R_f$ in parallel with $C$ is connected in order to increase the stability. In fact, the input signal will be integrated properly if the time period $T$ of the signal is larger than or equal to $R_fC$.

In a practical integrator circuit, a resistor $R_2$ is included between the non-inverting terminal and ground to reduce the component of bias current flowing towards $C$. A good choice of selection of $R_2$ is parallel combination of $R_1$ and $R_f$. 

![A practical integrator circuit](image_url)
A good low-cost OpAmp suitable for building an integrator circuit:

National Semiconductor

LF351
Wide Bandwidth JFET Input Operational Amplifier

General Description
The LF351 is a low cost high speed JFET input operational amplifier with an internally trimmed input offset voltage (BI-FET II™ technology). The device requires a low supply current and yet maintains a large gain bandwidth product and a fast slope rate. In addition, well matched high voltage JFET input devices provide very low input bias and offset currents. The LF351 is pin compatible with the standard LM741 and uses the same offset voltage adjustment circuitry. This feature allows designers to immediately upgrade the overall performance of existing LM741 designs.

The LF351 may be used in applications such as high speed integrators, fast D/A converters, sample-and-hold circuits and many other circuits requiring low input offset voltage, low input bias current, high input impedance, high slew rate and wide bandwidth. The device has low noise and offset voltage drift, but for applications where these requirements are critical, the LF356 is recommended. If maximum supply current is important, however, the LF351 is the better choice.

Typical Connection

![Typical Connection Diagram]