The Exponential Form Fourier Series

Recall that the compact trigonometric Fourier series of a periodic, real signal \( f(t) \) with frequency \( \omega_0 \) is expressed as

\[
f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)
\]

Employing the Euler’s formula-based representation \( \cos(x) = \frac{1}{2} (e^{jx} + e^{-jx}) \) we may express the \( n \)th term in the above formulation as

\[
C_n \cos(n\omega_0 t + \theta_n) = \frac{1}{2} C_n [e^{j(n\omega_0 t + \theta_n)} + e^{-j(n\omega_0 t + \theta_n)}]
\]

\[
= \left( \frac{C_n}{2} e^{j\theta_n} \right) e^{jn\omega_0 t} + \left( \frac{C_n}{2} e^{-j\theta_n} \right) e^{-jn\omega_0 t}
\]

Let us define the following two complex constants, which will be referred to as the complex Fourier series coefficients,

\[
D_n = \frac{C_n}{2} e^{j\theta_n} \text{ and its complex conjugate } D_n^* = \frac{C_n}{2} e^{-j\theta_n}
\]

We will use the following notation to refer to the magnitude and angle of those constants:

\[
D_n = |D_n| \angle D_n = \frac{C_n}{2} e^{j\theta_n} \rightarrow |D_n| = \frac{C_n}{2} \text{ and } \angle D_n = \theta_n \quad n = 1, 2, 3, ...
\]

\[
D_{-n} \triangleq D_n^* = |D_{-n}| \angle D_{-n} = \frac{C_n}{2} e^{-j\theta_n} \rightarrow |D_{-n}| = \frac{C_n}{2} \text{ and } \angle D_{-n} = -\theta_n
\]
Note that for any real \( f(t) \) we would have \( |D_n| = |D_{-n}| = \frac{c_n}{2} \) and \( \angle D_n = -\angle D_{-n} = \theta_n \). Also, the dc component (signal average) is \( D_0 = C_0 \) which is a real-valued constant.

Therefore, we may express the exponential Fourier series representation of \( f(t) \) as (this is also known as the Fourier synthesis equation)

\[
f(t) = D_0 + \sum_{n=-\infty}^{+\infty} D_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{+\infty} D_n e^{jn\omega_0 t}
\]

Alternatively, the \( D_n \) coefficients can be computed directly using the integral (which is also known as the Fourier analysis equation)

\[
D_n = \frac{1}{T_0} \int_{T_0} f(t)e^{-jn\omega_0 t} dt \quad n = 0, \pm 1, \pm 2, \pm 3, ...
\]

In some situations, the \( D_0 \) term must be computed as

\[
D_0 = \lim_{n \to 0} \frac{1}{T_0} \int_{T_0} f(t)e^{-jn\omega_0 t} dt
\]

The above solution for the \( D_n \) coefficients can be shown to be the one that is optimal in the sense of minimizing the total squared-error objective function,

\[
I(D_n) = \int_0^{T_0} \left[ f(t) - \sum_{n=-\infty}^{+\infty} D_n e^{jn\omega_0 t} \right]^2 dt
\]

where \( \omega_0 \) is the frequency of the real, periodic signal \( f(t) \).
The above optimization problem is equivalent to minimizing the following objective function [based on the compact Fourier series expansion]

\[
I(C_n, \theta_n) = \int_{T_0} \left[ f(t) - \sum_{n=0}^{\infty} C_n \cos(n\omega_0 t + \theta_n) \right]^2 dt
\]

(Refer to the last few slides of Lecture 14 for an example)

The following table lists the coefficients associated with the three Fourier series representations.

<table>
<thead>
<tr>
<th>Trigonometric Series</th>
<th>Compact Series</th>
<th>Exponential Series</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a_0, a_n, b_n)</td>
<td>(C_0, C_n, \theta_n)</td>
<td>(D_0, D_n)</td>
</tr>
<tr>
<td>(n = 1, 2, 3, \ldots)</td>
<td>(n = 1, 2, 3, \ldots)</td>
<td>(n = \pm 1, \pm 2, \ldots)</td>
</tr>
</tbody>
</table>

For a real and periodic \(f(t)\), we have the following formulas for the complex coefficients in terms of the standard trigonometric Fourier series coefficients:

\[
D_n = |D_n| e^{j\theta_n} = Re\{D_n\} + j Im\{D_n\}
\]

\[
|D_n| = \frac{1}{2} C_n = \frac{1}{2} \sqrt{a_n^2 + b_n^2} \quad n = 1, 2, \ldots
\]

\[
D_0 = C_0 = a_0
\]

\[
\angle D_n = \theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right) \quad n = 1, 2, 3, \ldots
\]

\[
|D_{-n}| = |D_n| \quad \text{and} \quad \angle D_{-n} = -\theta_n
\]
The trigonometric Fourier series coefficients can be determined from the complex coefficients as follows,

\[ a_0 = D_0 \]
\[ a_n = 2|D_n| \cos(\theta_n) = 2\text{Re}\{D_n\} \]
\[ b_n = -2|D_n| \sin(\theta_n) = -2\text{Im}\{D_n\} \]

Similarly, the compact coefficients can be determined from the trigonometric (or complex) coefficients as follows,

\[ C_n = \sqrt{a_n^2 + b_n^2} = 2|D_n| \quad n = 1, 2, 3, \ldots \]
\[ \theta_n = \tan^{-1} \left( -\frac{b_n}{a_n} \right) = \angle D_n \]

The following table summarizes all transformations between the three Fourier series representations.
Exponential Fourier Series Spectra

The exponential Fourier series spectra of a periodic signal \( f(t) \) are the plots of the magnitude and angle of the complex Fourier series coefficients.

Let \( f(t) \) be a real, periodic signal (with frequency \( \omega_0 \)).

Then \(|D_n| = |D_{-n}| \) and \( \theta_{-n} = -\theta_n \)

![Diagram showing even and odd functions of Fourier series coefficients](image)
Power Calculations and Perseval’s Theorem

\[ P_f = \frac{1}{T_0} \int_{T_0} |f(t)|^2 dt \]

If \( f(t) \) is real, then \( |f(t)|^2 = f^2(t) \). According to Parseval’s Theorem, the average power of a sum of harmonic signals can be computed as

\[ P_f = C_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} C_n^2 = D_0^2 + \frac{1}{2} \sum_{n=1}^{+\infty} 4|D_n|^2 \]

\[ \therefore P_f = \frac{1}{T_0} \int_{T_0} f^2(t) dt = D_0^2 + 2 \sum_{n=1}^{\infty} |D_n|^2 \]

Symmetry Considerations

- \( f(t) \) is even \( \rightarrow b_n = 0 \rightarrow Im\{D_n\} = 0 \rightarrow D_n \) is real.
- \( f(t) \) is odd \( \rightarrow a_n = 0 \rightarrow Re\{D_n\} = 0 \rightarrow D_n \) is pure imaginary.
- Half-wave symmetry: \( f\left(t \pm \frac{T_0}{2}\right) = -f(t) \rightarrow a_n, b_n = 0 \) for \( n \) even \( \rightarrow D_n = 0 \), for all even \( n \).

Example. Consider the following periodic signal represented as an exponential Fourier series,

\[ f(t) = \frac{1}{2} + \frac{j}{2\pi} \sum_{n=\infty}^{\infty} \frac{(-1)^n - 1}{n} e^{j2\pi nt} \]

Answer the following questions:
- Determine the fundamental frequency $\omega_0$.
- Determine the symmetries in $f(t)$.
- Is there a jump discontinuity in $f(t)$?
- Plot the FS approximation of $f(t)$ using Mathcad. Consider only the constant term and the first 5 harmonics.

Solution:

$$e^{j2\pi nt} = e^{j\omega_0 t} \rightarrow \omega_0 = 2\pi.$$

$$D_n = j\left[\frac{(-1)^n-1}{2\pi n}\right], \text{ pure imaginary } \rightarrow f(t) \text{ is odd.}$$

$$D_n = \begin{cases} 0 & n \text{ even} \\ \frac{j}{n\pi} & n \text{ odd} \end{cases} \rightarrow f(t) \text{ has half-wave symmetry}$$

$$|D_n| = \frac{1}{n\pi} = O\left(\frac{1}{n}\right) \rightarrow f(t) \text{ has a jump discontinuity.}$$

The following is a Mathcad generated plot for the truncated (dc plus the first five harmonics) Fourier series approximation of $f(t)$. (Note how the sum is separated into two sub-sums with the dc term extracted.)

$$j := \sqrt{-1}$$

$$f(t) := \frac{1}{2} + \frac{j}{2\pi} \left[ \sum_{n=-5}^{-1} \frac{(-1)^n-1}{n} \cdot e^{j2\pi nt} + \sum_{n=1}^{5} \frac{(-1)^n-1}{n} \cdot e^{j2\pi nt} \right]$$

![Mathcad generated plot](image-url)
Example. Exponential Fourier series spectra for simple signals. Determine the exponential Fourier series representation and plot its spectra for the signals:

a. \( f(t) = \cos(\omega_0 t) \)
b. \( f(t) = \sin(\omega_0 t) \)

a. By inspection of \( f(t) \) we have \( C_0 = 0, \ C_1 = 1, \) and \( \theta_1 = 0 \). Therefore, \( D_0 = C_0 = 0 \)

\[
|D_1| = \frac{1}{2} C_1 = \frac{1}{2}, \quad \angle D_1 = \theta_1 = 0
\]

\[
|D_{-1}| = |D_1| = \frac{1}{2}, \quad \angle D_{-1} = -\theta_1 = 0
\]

Leading to the spectra:

Alternatively: \( f(t) = \cos(\omega_0 t) = \frac{1}{2} e^{j\omega_0 t} + \frac{1}{2} e^{-j\omega_0 t} \). Then,

\[
D_0 = 0, \quad |D_1| = |D_{-1}| = \frac{1}{2}
\]

\[
\theta_1 = \theta_{-1} = 0
\]

Since \( f(t) \) is even \( \rightarrow D_n \) is real valued (here, \( D_n = \frac{1}{2} \))
b. First, we need to express the signal in terms of the cosine function,

\[ f(t) = \sin(\omega_0 t) = \cos\left(\omega_0 t - \frac{\pi}{2}\right), \quad C_1 = 1, \quad \theta_1 = -\frac{\pi}{2} \]

\[ D_0 = C_0 = 0 \]

\[ |D_1| = |D_{-1}| = \frac{1}{2} \]

\[ \angle D_1 = \theta_1 = -\frac{\pi}{2} \]

\[ \angle D_{-1} = -\theta_1 = \frac{\pi}{2} \]
**Example.** Consider the following signal

\[ f(t) = e^{-\frac{t}{2}} \text{ with } T_0 = \pi \text{ or } \omega_0 = \frac{2\pi}{T_0} = 2 \]

Express \( f(t) \) as an exponential Fourier series,

\[ f(t) = \sum_{n=-\infty}^{+\infty} D_n e^{j\omega_0 t} \]

From earlier analysis, we have

\[ a_0 = \frac{2}{\pi} \left(1 - e^{-\frac{\pi}{2}}\right); \quad a_n = \frac{2}{\pi} \left(1 - e^{-\frac{\pi}{2}}\right) \left(\frac{2}{1 + 16n^2}\right) \]

\[ b_n = \frac{2}{\pi} \left(1 - e^{-\frac{\pi}{2}}\right) \left(\frac{8n}{1 + 16n^2}\right) \]

\[ D_0 = C_0 = a_0 = \frac{2}{\pi} \left(1 - e^{-\frac{\pi}{2}}\right) \]

\[ C_n = \sqrt{a_n^2 + b_n^2} = \frac{2}{\pi} \left(1 - e^{-\frac{\pi}{2}}\right) \left(\frac{2}{\sqrt{1 + 16n^2}}\right) \]

\[ \theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right) = -\tan^{-1}(4n) \]

\[ |D_n| = |D_{-n}| = \frac{1}{2} C_n = \frac{2}{\pi} \left(1 - e^{-\frac{\pi}{2}}\right) \sqrt{1 + 16n^2}, \quad n = 1, 2, 3 \ldots \]
\[ \angle D_n = \theta_n = -\tan^{-1}(4n) \]
\[ \angle D_{-n} = -\theta_n = +\tan^{-1}(4n) \]

Therefore,
\[ D_n = |D_n|e^{j\theta_n} = \frac{2}{\pi} \frac{\left(1 - e^{-\frac{\pi}{2}}\right)}{\sqrt{1 + 16n^2}} e^{-j\tan^{-1}(4n)} \]

leading to the exponential Fourier series representation of \( f(t) \),
\[
    f(t) = \sum_{n=-\infty}^{+\infty} D_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{+\infty} \frac{2}{\pi} \frac{\left(1 - e^{-\frac{\pi}{2}}\right)}{\sqrt{1 + 16n^2}} e^{-j\tan^{-1}(4n)} e^{j2nt}
\]

Or,
\[
    f(t) = \frac{2}{\pi} \left(1 - e^{-\frac{\pi}{2}}\right) \sum_{n=-\infty}^{+\infty} \frac{1}{\sqrt{1 + 16n^2}} e^{j[2nt - \tan^{-1}(4n)]}
\]

**Your Turn:** Show that the above series has the alternative representation,
\[
    f(t) = \frac{2}{\pi} \left(1 - e^{-\frac{\pi}{2}}\right) \sum_{n=-\infty}^{+\infty} \frac{1 - j4n}{1 + 16n^2} e^{j2nt}
\]

[Note: \( e^{-j\tan^{-1}(4n)} = \cos(\tan^{-1} 4n) - j\sin(\tan^{-1} 4n) \)]

[Hint: \( \cos(\tan^{-1} x) = \frac{1}{\sqrt{1+x^2}} \) and \( \sin(\tan^{-1} x) = \frac{x}{\sqrt{1+x^2}} \)]
Alternatively, we may solve for $D_n$ using the fundamental integral,

$$D_n = \frac{1}{T_0} \int_{T_0} f(t) e^{-jn\omega_0 t} \, dt \quad n = 0, \pm 1, \pm 2 \ldots$$

$$D_n = \frac{1}{\pi} \int_{0}^{\pi} e^{-t/2} e^{-j2nt} \, dt$$

$$D_n = \frac{1}{\pi} \int_{0}^{\pi} e^{-\left(\frac{1}{2}+j2n\right)t} \, dt = \frac{1}{\pi} \left(\frac{-2}{1+j4n}\right) \left[e^{-\frac{\pi}{2}} e^{-j2n\pi} - e^0\right]$$

Note: $e^{-j2n\pi} = \cos(-2n\pi) + j\sin(-2n\pi) = 1, \forall_n$

$$D_n = \frac{2 \left(1 - e^{-\frac{\pi}{2}}\right)}{\pi(1 + j4n)} = \frac{2(1 - j4n) \left(1 - e^{-\frac{\pi}{2}}\right)}{\pi(1 + 16n^2)}$$

$$Re\{D_n\} = \frac{2 \left(1 - e^{-\frac{\pi}{2}}\right)}{\pi(1 + 16n^2)} = \frac{a_n}{2}$$

$$Im\{D_n\} = \frac{-8n \left(1 - e^{-\frac{\pi}{2}}\right)}{\pi(1 + 16n^2)} = -\frac{b_n}{2}$$

Therefore, the expansion is given by,

$$f(t) = \frac{2}{\pi} \left(1 - e^{-\frac{\pi}{2}}\right) \sum_{n=-\infty}^{+\infty} \frac{1 - j4n}{1 + 16n^2} e^{j2nt}$$

Note that the convergence of the coefficients is of order $|D_n| = O\left(\frac{1}{n}\right)$.

Setting $n = 0$, we obtain $D_0 = \frac{2}{\pi} \left(1 - e^{-\frac{\pi}{2}}\right)$ which is the signal’s average.
Verification of the answer by plotting (employing \(-8 \leq n \leq 8\)):

\[
j := \sqrt{-1}
\]

\[
f(t) = \frac{2}{\pi} \left(1 - \frac{-\pi}{2} \right) \sum_{n = -8}^{8} \frac{(1 - j4n)}{1 + (16n)^2} e^{j2n t}
\]

---

**Mathcad Computation of Various Fourier Series Representations**

\[
T := \pi \quad n := 6
\]

\[
f(t) := e^{-\frac{t}{2}}
\]

\[
\omega := \frac{2\pi}{T} \quad j = \sqrt{-1}
\]

\[
D1(k) := \frac{1}{T} \left( \int_{0}^{T} f(t) e^{-j\omega k t} dt \right) \rightarrow \frac{1}{\pi} \left[ \frac{-2}{1 + 4i k} \exp \left( \frac{-1}{2} \pi \left( 1 + 4i k \right) \right) + \frac{2}{(1 + 4i k)} \right]
\]

\[
k := -n .. n
\]

\[
D_k := \lim_{m \to k} D1(m) \text{ complex} \rightarrow \left( \frac{-2}{577} + \frac{2}{577} \exp \left( \frac{1}{2} \pi \right) \right) \exp \left( -\frac{1}{2} \pi \right) + i \left( \frac{48}{577} - \frac{48}{577} \exp \left( \frac{1}{2} \pi \right) \right) \exp \left( -\frac{1}{2} \pi \right)
\]

\[
f(t) := \sum_{k = 0}^{2n} D_k \cdot e^{j\omega(k-n) t}
\]

\[
g(t) := f(t) \cdot (\Phi(t) - \Phi(t - \tau))
\]

For Mathcad representation convenience, note how the \(D_n\) coefficients’ subscripts have been shifted by \(n\), from \([-n,n]\) to \([0,2n]\) in the above worksheet.
Spectra

Complex Coefficients $D(n\geq0)$:  

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
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<th></th>
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</thead>
<tbody>
<tr>
<td>0</td>
<td>0.504</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.030-0.119</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>7.758-10-3-0.062</td>
<td></td>
<td></td>
</tr>
<tr>
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<td>3.478-10-3-0.042</td>
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<td></td>
</tr>
<tr>
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<td>1.962-10-3-0.031</td>
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<td></td>
</tr>
<tr>
<td>5</td>
<td>1.258-10-3-0.025</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>8.74-10-4-0.021</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Amplitudes $C(n)$:

$C = \begin{pmatrix} 0.504 \\ 0.245 \\ 0.125 \\ 0.084 \\ 0.063 \\ 0.05 \\ 0.042 \end{pmatrix}$

Phase $\theta(n)$ (rad/Degrees):

$\theta_{rad} = \begin{pmatrix} 0 \\ -1.326 \\ -1.446 \\ -1.488 \\ -1.508 \\ -1.521 \\ -1.529 \end{pmatrix}$  

$\theta_{deg} = \begin{pmatrix} 0 \\ -75.964 \\ -82.875 \\ -85.236 \\ -86.424 \\ -87.138 \\ -87.614 \end{pmatrix}$
Note: The horizontal axis represents frequency. The actual harmonic frequencies in the above spectra are $n\omega_0 = 2n$, $n = 0, \pm 1, \pm 2, ...$

**Trigonometric Fourier Series Coefficients**

\[
\begin{align*}
a_k &= 2 \cdot \text{Re}(D_k) & b_k &= -2 \cdot \text{Im}(D_k) \\
a_0 &= \text{Re}(D_0)
\end{align*}
\]

\[
a = \begin{pmatrix}
0.504 \\
0.059 \\
0.016 \\
6.956 \times 10^{-3} \\
3.924 \times 10^{-3} \\
2.515 \times 10^{-3} \\
1.748 \times 10^{-3}
\end{pmatrix}
\]

\[
b = \begin{pmatrix}
0 \\
0.237 \\
0.124 \\
0.083 \\
0.063 \\
0.05 \\
0.042
\end{pmatrix}
\]

**Exact Power of original signal**

\[
\frac{1}{T} \int_0^T f(t)^2 \, dt = 0.305 \\
(D_0)^2 + 2 \sum_{k=1}^{n} (|D_k|)^2 = 0.3
\]
**Example.** Consider the following square wave. Determine the $D_n$ coefficients.

\[ D_0 = f_{ave}(t) = \frac{A}{2} \] (by inspection of the plot).

If we add a bias of $-\frac{A}{2}$ to $f(t)$ we obtain the signal $f_1(t)$ that is plotted below.

Symmetry consideration:

$f_1(t)$ is odd and it has half-wave symmetry. Then, the complex coefficients of this signal, as well as those of the unbiased signal $f(t)$, have the following properties (why?):

$D_n$ is pure imaginary and $D_n = 0$ for all even $n$.

Also, $f(t)$ has a jump discontinuity $\rightarrow |D_n| = O\left(\frac{1}{n}\right)$.
Solving for the $D_n$ coefficients of $f(t)$ by direct integration:

$$D_n = \frac{1}{2} \left[ \int_0^1 Ae^{-jn\pi t} \, dt + \int_1^2 0 \, dt \right] = \frac{1}{2} \int_0^1 Ae^{-jn\pi t} \, dt$$

$$D_n = \frac{A}{2} \left( \frac{1}{-jn\pi} \right) e^{-jn\pi} \bigg|_0^1 = j \frac{A}{2\pi n} [e^{-jn\pi} - 1]$$

Since $e^{-jn\pi} = \begin{cases} +1 & n \text{ even} \\ -1 & n \text{ odd} \end{cases}$; then $D_n = j \frac{A}{2\pi n} [(-1)^n - 1], n \neq 0$

We may write

$$D_n = \begin{cases} -j \frac{A}{\pi n} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

The dc term, $D_0$, is obtained by setting $n = 0$ (we know that $D_0 = \frac{A}{2}$)

$$D_0 = j \frac{A}{2\pi n} [e^{-jn\pi} - 1] = 0$$

So, we employ L’Hospital’s rule:

$$D_0 = \lim_{n \to 0} D_n = \lim_{n \to 0} j \frac{A}{2\pi n} [e^{-jn\pi} - 1]$$

$$= \frac{ja}{2\pi} \lim_{n \to 0} \frac{d}{dn} \left( e^{-jn\pi} - 1 \right) = \frac{ja}{2\pi} \lim_{n \to 0} \left( \frac{-jn e^{-jn\pi}}{1} \right) = \frac{A}{2}$$

Therefore, $f(t)$ can now be expressed as

$$f(t) = \frac{A}{2} + \sum_{n=-\infty}^{+\infty} \left( \frac{-ja}{n\pi} \right) e^{jn\pi t}$$
Mathcad verification ($A = 1$):

\[
T := 2 \quad n := 8
\]

\[
f(t) := (\Phi(t) - \Phi(t - 1))
\]

\[
\omega := \frac{2\pi}{T} \quad j \equiv \sqrt{-1}
\]

\[
D_l(k) := \frac{1}{T} \int_{0}^{T} f(t) e^{-j\omega \cdot k \cdot t} \, dt
\]

\[
k := -n \ldots n
\]

\[
D_{k+n} := \left( \lim_{m \to k} D_1(m) \right) \to 0
\]

**Complex coefficients:**

<table>
<thead>
<tr>
<th>$m \to 0$</th>
<th>$m \to 1$</th>
<th>$m \to 2$</th>
<th>$m \to 3$</th>
<th>$m \to 4$</th>
<th>$m \to 5$</th>
<th>$m \to 6$</th>
<th>$m \to 7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{2}$</td>
<td>$-i$</td>
<td>$\frac{1}{\pi}$</td>
<td>$0$</td>
<td>$-\frac{1}{3} \cdot i$</td>
<td>$\frac{3}{\pi}$</td>
<td>$0$</td>
<td>$-\frac{1}{7} \cdot i$</td>
</tr>
</tbody>
</table>

**Magnitude:**

<table>
<thead>
<tr>
<th>$m \to 0$</th>
<th>$m \to 1$</th>
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<th>$m \to 3$</th>
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<th>$m \to 7$</th>
</tr>
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<tbody>
<tr>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{\pi}$</td>
<td>$0$</td>
<td>$\frac{1}{3} \cdot \pi$</td>
<td>$0$</td>
<td>$\frac{1}{5} \cdot \pi$</td>
<td>$0$</td>
<td>$\frac{1}{7} \cdot \pi$</td>
</tr>
</tbody>
</table>

**Angle:**

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<th>$m \to 6$</th>
<th>$m \to 7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0$</td>
<td>$-\frac{1}{2} \cdot \pi$</td>
<td>$0$</td>
<td>$-\frac{1}{2} \cdot \pi$</td>
<td>$0$</td>
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</tbody>
</table>
\[ f_l(t) := \sum_{k=0}^{2n} D_k e^{j\omega(k-n)t} \]

\[ g(t) := f(t) \left( \Phi(t) - \Phi(t-T) \right) \]

**Reconstructed signal (n harmonics):**

\[ g(t-T) + g(t) + g(t+T) \]

\[ f_l(t) \]
Note: The horizontal axis represents frequency. The actual harmonic frequencies in the above spectra are $n\omega_0 = n\pi$.

Exact Power of original signal          Power in dc + first
\[ \frac{1}{T} \int_0^T f(t)^2 \, dt = 0.5 \] \[ (D_0)^2 + 2 \sum_{k=1}^{n} (|D_k|)^2 = 0.487 \]
Your turn: Consider the periodic signal \( f(t) \),

\[
f(t) = t^2, -\pi < t < \pi \quad \text{and} \quad f(t \pm 2k\pi) = f(t), k = 0,1,2, \ldots
\]

(a) What are the symmetries in the signal?

(b) Predict the convergence rate of the Fourier series coefficients, \( D_n \).

(c) Find (directly) the exponential Fourier series for \( f(t) \).

(d) Compare the signal’s exact power to that obtained using the dc and first 5 harmonic terms.

(e) Plot the signal’s spectra.

(f) Verify your work employing the provided Mathcad exponential Fourier series worksheet.

Your turn: Consider the signal \( f(t) \),

\[
f(t) = |\sin(\pi t)|
\]

(a) What are the symmetries in the signal?

(b) Predict the convergence rate of the Fourier series coefficients \( D_n \).

(c) Find (directly) the exponential Fourier series for \( f(t) \).

(d) Compare the signal’s exact power to that obtained using the dc and first 5 harmonic terms.

(e) Plot the signal’s spectra.

(f) Verify your work employing the provided Mathcad exponential Fourier series worksheet.

Ans. (c)

\[
f(t) = -\frac{2}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{4n^2 - 1} e^{j2\pi nt}
\]
**Your turn:** Consider the following (even symmetry) pulse train signal with period $T$, pulse width $\tau$ and pulse height $A$.

![Image of pulse train signal](image)

Determine the signal’s compact trigonometric and exponential Fourier series representations. Note: For verification purposes, and for the special case of $T = 0.1$, $\tau = 0.025$ and $A = 1$ the series is given by

$$f(t) = \frac{1}{4} + \frac{\sqrt{2}}{\pi} \cos(20\pi t) + \frac{1}{\pi} \cos(40\pi t) + \frac{\sqrt{2}}{3\pi} \cos(60\pi t) + \cdots$$

Next, set $A = \frac{1}{\tau}$ and determine the exponential Fourier series in the limit $\tau \to 0$.

Sketch the resulting signal. Revisit your answers to this problem after you solve the next “your turn” problem.

**Your turn:** Consider the following (even symmetry) impulse train signal with period $T$, $\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$.

![Image of impulse train signal](image)
(a) Show that the signal’s (exponential) Fourier series expansion is given by

\[ \delta_T(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{j2\pi n t/T} \]

(b) Determine the signal’s compact trigonometric Fourier series and plot its spectra.

(c) Plot the signal \( \delta_T(t) \) for \( T = 1 \) using 10 terms. Repeat for 100 terms.

(d) How would you characterize the rate of convergence of this series?

**Example.** Consider a 1KHz, 50% duty-cycle square wave with zero-average and amplitude of 2 Volts. Assume that the first cycle of the signal starts at \( t = 0 \); this leads to an odd symmetry signal. Plot the sum of the first three harmonics, and then plot the sum of the third, fourth and fifth harmonics.

Mathcad can be used to obtain (numerically) the exponential Fourier series for this signal, as follows:

\[ T := 0.001 \]

\[ \omega := \frac{2\pi}{T} \quad j = \sqrt{-1} \]

\[ D1(k) := \frac{1}{T} \left( \int_{0}^{T/2} 2e^{-j\omega\cdot k \cdot t} \, dt + \int_{T/2}^{T} -2e^{-j\omega\cdot k \cdot t} \, dt \right) \]

\[ |D_k| \]

\[ k \]

\[ -6 \quad -4 \quad -2 \quad 0 \quad 2 \quad 4 \quad 6 \]
Note that the even harmonics are missing because of the odd symmetry of $f(t)$.

The following is a plot of the signal $f(t)$ and the sum of the first and third harmonics, $f_1(t)$. [Mathcad’s array origin is set to -5.]

$$f_1(t) := \sum_{k = -3}^{3} D_k \cdot e^{j \omega \cdot k \cdot t}$$

The following plot displays the sum of the first and third harmonics $f_1(t)$ (biased by +4, for the sake of display clarity), along with the signal $f_2(t)$ that is generated from adding the third and fifth harmonics (recall that the forth harmonic is zero).

$$f_2(t) := \sum_{k = -5}^{-3} D_k \cdot e^{j \omega \cdot k \cdot t} + \sum_{k = 3}^{5} D_k \cdot e^{j \omega \cdot k \cdot t}$$