Linear Time-Invariant Discrete-Time (LTID) System Analysis

Consider a linear discrete-time system. We are interested in solving for the complete response $y[k]$ given the difference equation governing the system, its associated initial conditions and the input $f[k]$.

$$f[k] \rightarrow \text{system} \rightarrow y[k] \rightarrow \text{Difference equation} \rightarrow \text{solve for } y[k], k \geq 0$$

Advance Formulation \quad Delay Formulation

It will be shown that $y[k]$ is a sequence of numbers that can be obtained explicitly by simple recursion (iteration). Alternatively, an analytic solution of the linear difference equation can be obtained based on the Z-transform method. The analytic solution can also be obtained based on a systematic time-domain method, as covered in the next lecture.

There are two equivalent formulations for a difference equation: The advance operator formulation and the delay operator formulation. The following analysis illustrates how to convert one formulation to the other.
**Delay Operator Formulation**

The following is an example of a delay formulation, second-order linear difference equation. Here, the signal $y[k]$ appears right shifted,

\[
y[k] - 5y[k - 1] + 6y[k - 2] = f[k - 1] - 5f[k - 2] \quad (1)
\]

with the initial conditions $y[-1] = 2$ and $y[-2] = 1$. Let the input signal be the causal signal $f[k] = (3k + 5)u[k]$. Solve for $y[k], k \geq 0$

**Advance Operator Formulation**

The advance operator formulation for the above difference equation can be obtained by simply shifting the difference equation so that the largest difference term $y[k - 2]$ becomes $y[k]$. Which requires replacing $k$ by $k + 2$ throughout the equation, to obtain:

\[
y[k + 2] - 5y[k + 1] + 6y[k] = f[k + 1] - 5f[k] \quad (2)
\]

Next, we need to establish a set of appropriate initial conditions, $y[0]$ and $y[1]$. This can be done by setting $k$ to 0 and then to 1 in Equation (1) and solving for $y[0]$ and $y[1]$, respectively, as follows.

Set $k = 0$ in Eqn. (1) to obtain $y[0]$,

\[
y[0] = 5y[-1] - 6y[-2] + f[-1] - 5f[-2]
\]
\[
y[0] = 5(2) - 6(1) + 0 - 0 = 10 - 6 \rightarrow y[0] = 4
\]
Set $k = 1$ in Eqn. (1) to obtain $y[1]$, 

$$y[1] = 5y[0] - 6y[-1] + f[0] - 5f[-1]$$

$$y[1] = 5(4) - 6(2) + 5 - 0 = 13 \rightarrow y[1] = 13$$

It is important to note that when we derived the initial conditions for this system, one of them ($y[1]$) depended on the input value $f[0]$. So we say that the initial conditions are *tainted* by the input. When that occurs, the advance formulation should be avoided if we are looking for decoupled solutions of the form $y[k] = y_{zs}[k] + y_{zi}[k]$. This will be illustrated through an example later in this lecture.

**Analysis Methods for Computing $y[k]$**

- Iterative method (numerical solution; also works for non-linear systems)
- Z-transform method; allows for finding $y_{zs}[k]$ and $y_{zi}[k]$.  
- Time-domain convolution for finding $y_{zs}[k]$. (Refer to the last section.)
- Time-domain analysis. (Refer to the classical method of solution of difference equations covered in Lecture 22.)

Solve the above advance-formulation, second-order linear difference equation using the iterative method. First, we solve for the highest delay term in the difference equation to arrive at the *recursion formula*

$$y[k + 2] = 5y[k + 1] - 6y[k] + [3(k + 1) + 5]u[k + 1] - 5(3k + 5)u[k]$$

and then start iterating with $y[0] = 4$ and $y[1] = 13$, as is shown below.

$k = 0$: \[ y[2] = 5y[1] - 6y[0] + (3 + 5)(1) - 5(0 + 5)(1) \]

$$y[2] = 5(13) - 6(4) + 8 - 25 = 24$$
\[ y[3] = 5(24) - 6(13) + 11 - 40 = 13 \]

Repeat for $k = 2, 3, 4, \ldots$ to generate the solution sequence ($k \geq 0$)
\[ y[k] = 4, 13, 24, 13, -120, \ldots \]

Next, we illustrate the iterative solution applied to the delay formulation equation, Eqn. (1). First, we obtain the recursion formula for $y[k]$ as,
\[ y[k] = 5y[k - 1] - 6y[k - 2] + (3k + 2)u[k - 1] - 5(3k - 1)u[k - 2] \]
and start iterating with $y[-1] = 2$ and $y[-2] = 1$ as follows,

$k = 0$: \[ y[0] = 5y[-1] - 6y[-2] + 2u[-1] - 5(-1)u[-2] \]
\[ y[0] = 5(2) - 6(1) + 0 + 0 \]
\[ y[0] = 4 \]

$k = 1$: \[ y[1] = 5y[0] - 6y[-1] + 5u[0] - 5(2)u[-1] \]
\[ y[1] = 5(4) - 6(2) + 5 - 0 \]
\[ y[1] = 13 \]

Repeat for $k = 2, 3, 4, \ldots$ to generate the solution sequence ($k \geq 0$)
\[ y[k] = 4, 13, 24, 13, -120, \ldots \]

This sequence is identical to the one obtained earlier using the advance formulation.
Mathcad numerical solution for the advance operator difference equation (note: press ‘[‘ to input the subscript):

\[ y_0 := 4 \quad y_1 := 13 \quad f(k) := (3k + 5)\Phi(k) \]

\[ k := 0..6 \]

\[ y_{k+2} = 5y_{k+1} - 6y_k + f(k + 1) - 5f(k) \]

\[ y_k = \begin{bmatrix} 4  \\ 13  \\ 24  \\ 13  \\ -120  \\ -731  \\ -3\times10^3 \end{bmatrix} \]

Mathcad numerical solution for the delay operator difference equation

\[ y_{-1} := 2 \quad y_{-2} := 1 \quad f(k) := (3k + 5)\Phi(k) \]

\[ k := 0..6 \]

\[ y_k := 5y_{k-1} - 6y_{k-2} + f(k - 1) - 5f(k - 2) \]

\[ y_k = \begin{bmatrix} 4  \\ 13  \\ 24  \\ 13  \\ -120  \\ -731  \\ -3\times10^3 \end{bmatrix} \]
Note how the above Mathcad solution automatically performs the required iterations to compute the sequence (vector) $y$.

Mathcad allows the user to redefine the array origin. For the above delay operator recursion we had to set the origin at $-2$ in the Math Options menu, as illustrated below. (For the advance formulation solution the array origin has to be set to 0.)
Solving for the Complete Response \( y[k] = y_{zs}[k] + y_{zi}[k] \) using the Z-Transform Method

The Z-transform method can be applied to the difference equation to (analytically) generate the complete response. However, if we intend to use it to generate the decoupled response \( y[k] = y_{zs}[k] + y_{zi}[k] \), we have to make sure that the initial conditions we use are not dependent on the input signal.

As pointed out earlier, one of the advance formulation initial conditions (for our example system) is dependent on the input. Therefore, for that advance formulation system the Z-transform method can be applied to generate \( y[k] \), but not \( y_{zs}[k] \) and \( y_{zi}[k] \). On the other hand, the Z-transform method can be used to generate \( y[k] = y_{zs}[k] + y_{zi}[k] \) for the delay formulation system.

**Example.** Z-transform analysis for the delay formulation system:

\[
y[k] - 5y[k - 1] + 6y[k - 2] = f[k - 1] - 5f[k - 2]
\]

\[
y[-1] = 2 , \ y[-2] = 1
\]

\[
f[k] = (3k + 5)u[k]
\]

We will find \( y_{zs}[k] \) first by Z-transforming the difference equation, assuming zero initial conditions, to obtain \( H(z) \) and then employ the formula,

\[
y_{zs}[k] = Z^{-1}\{H(z)F(z)\}
\]

But first, we must be clear about the meaning of the shifted signals \( y[k - 1] \) and \( y[k - 2] \). Do they mean \( y[k - 1]u[k] \) and \( y[k - 2]u[k] \) or \( y[k - 1]u[k - 1] \) and \( y[k - 2]u[k - 2] \)?
Well, since our Z-transform is one-sided it means that we are considering the situation for $k \geq 0$. So, every signal is assumed to be causal and is counted from $k = 0$. Therefore, the terms $y[k - m]$ mean $y[k - m]u[k]$, for integer $m$. Transforming (and using the right-shift property) we obtain

$$Y(z) - 5 \left( \frac{1}{z} Y(z) + y[-1] \right) + 6 \left( \frac{1}{z^2} Y(z) + \frac{1}{z} y[-1] + y[-2] \right)$$

$$= \left( \frac{1}{z} F(z) + f[-1] \right) - 5 \left( \frac{1}{z^2} F(z) + \frac{1}{z} f[-1] + f[-2] \right)$$

which, with the requirement that the initial conditions are zero (zero-state) and by noting that $f[k]$ is a causal signal $\{f[-1] = f[-2] = 0\}$, reduces to

$$\left(1 - \frac{5}{z} + \frac{6}{z^2}\right) Y_{zs}(z) = \left(\frac{1}{z} - \frac{5}{z^2}\right) F(z)$$

Next, multiplying the above equation by $z^2$, leads to

$$(z^2 - 5z + 6) Y_{zs}(z) = (z - 5) F(z)$$

which has the form $Q(z) Y_{zs}(z) = P(z) F(z)$. The transfer function is then,

$$H(z) \triangleq \frac{Y_{zs}(z)}{F(z)} = \frac{P(z)}{Q(z)}$$

which results in

$$H(z) = \frac{z-5}{(z-2)(z-3)} \rightarrow \text{The poles are at } z_1 = 2, z_2 = 3$$

The system is unstable since $|z_i| > 1$.

Before we proceed to solve for $y_{zs}[k]$, let us determine the unit-impulse response,

$$h[k] = Z^{-1}\{H(z)\}$$
First, perform partial fraction expansion on $\frac{H(z)}{z}$ (say, using Mathcad):

$$
\frac{z - 5}{z(z - 2)(z - 3)} \text{ convert, parfrac, } z \rightarrow \frac{-5}{6z} + \frac{3}{2(z - 2)} - \frac{2}{3(z - 3)}
$$

Next, multiply by $z$ and use the table of Z-transforms to find the inverse Z-transform:

$$
H(z) = -\frac{5}{6z} + \frac{3}{2(z - 2)} - \frac{2}{3(z - 3)}
$$

$$
h[k] = -\frac{5}{6} \delta[k] + \frac{3}{2} (2)^k u[k] - \frac{2}{3} (3)^k u[k]
$$

The zero-state response $y_{zs}[k]$ may now be obtained using convolution sums,

$$
y_{zs}[k] = h[k] * f[k], \text{ where } f[k] = 3k u[k] + 5u[k]
$$

[Your turn: Employ the Convolution Table for discrete-time signals, Page 590, to solve for $y_{zs}[k]$. Refer to the last section (last three slides of this lecture) for an example.]

Alternatively, we may use the inverse Z-transform to obtain $y_{zs}[k]$ as follows,

$$
y_{zs}[k] = Z^{-1}\{Y_{zs}(z)\} = Z^{-1}\{H(z)F(z)\}
$$

$$
y_{zs}[k] = Z^{-1} \left\{ \frac{z - 5}{(z - 2)(z - 3)} \frac{z(5z - 2)}{(z - 1)^2} \right\}
$$

where the following transform pair was used to obtain $F(z)$,
\[ f[k] = (3k + 5)u[k] \leftrightarrow F(z) = \frac{3z}{(z - 1)^2} + \frac{5z}{z - 1} = \frac{z(5z - 2)}{(z - 1)^2} \]

**Your turn:** Apply partial fraction expansion to \( \frac{Y_{zs}(z)}{z} \), multiply the result by \( z \), then employ the Z-transform Table to obtain the following result,

\[ y_{zs}[k] = \left( -\frac{35}{2} - 6k - \frac{13}{2} (3)^k + 24(2)^k \right) u[k] \]

Next, we solve for the zero-input response, \( y_{zi}[k] \).

\[
\begin{align*}
f[k] = 0 & \quad \text{I.C.} \quad y_{zi}[k] \\
y[-1] = 2, y[-2] = 1
\end{align*}
\]

Set \( f[k] = 0 \) in the difference equation,

\[ y[k] - 5y[k - 1] + 6y[k - 2] = 0 \]

Apply the Z-transform (include the contribution due to initial conditions),

\[ Y_{zi}(z) - 5 \left( \frac{1}{z} Y_{zi}(z) + y[-1] \right) + 6 \left( \frac{1}{z^2} Y_{zi}(z) + \frac{1}{z} y[-1] + y[-2] \right) = 0 \]

Rearrange terms so as to keep the equation in the form

\[ Q(z)Y_{zi}(z) + I(z) = 0 \]

\[
\left( 1 - \frac{5}{z} + \frac{6}{z^2} \right) Y_{zi}(z) + \left( -10 + \frac{12}{z} + 6 \right) = 0
\]

Multiply the equation by \( z^2 \),
\[
(z^2 - 5z + 6) Y_{zi}(z) + \left( -4z^2 + 12z \right) = 0
\]

Now we solve for \( Y_{zi}(z) \) in the form of a rational function to obtain,

\[
Y_{zi}(z) = \frac{-I(z)}{Q(z)} = \frac{4z^2 - 12z}{z^2 - 5z + 6} = \frac{4z(z - 3)}{(z - 2)(z - 3)} = \frac{4z}{z - 2}
\]

and readily observe that the solution is \( y_{zi}[k] = 4(2)^k u[k] \). It is interesting to note that the specific initial conditions that we used have only excited one of the natural modes of the system [due to the cancellation of the \((z - 3)\) term]. The complete (decoupled) response can now be expressed as,

\[
y[k] = y_{zs}[k] + y_{zi}[k]
\]

\[
y[k] = \left( -\frac{35}{2} - 6k - \frac{13}{2} (3)^k + 24(2)^k \right) u[k] + 4(2)^k u[k]
\]

\[ y_{zs} \quad \quad y_{zi} \]
Mathcad Verification

Delay Operator formlation:

\[ f(n) := (3 \cdot n + 5) \cdot \Phi(n) \]
\[ F(z) := f(n) \text{ ztrans, } n \rightarrow z \cdot \frac{(-2 + 5 \cdot z)}{(z - 1)^2} \]

Impulse respon:

\[ H(z) := \frac{z - 5}{z^2 - 5z + 6} \]
\[ H(z) \text{ invztrans, } z \rightarrow -\frac{5}{6} \cdot \text{Dirac}(n) - \frac{2}{3} \cdot 3^n + \frac{3}{2} \cdot 2^n \]

Zero state response:

\[ y_{zs}(n) := H(z) \cdot F(z) \text{ invztrans, } z \rightarrow -\frac{35}{2} - 6 \cdot n + 24 \cdot 2^n - \frac{13}{2} \cdot 3^n \]

Zero input response:

\[ y_{zi}(n) := \frac{4z \cdot (z - 3)}{z^2 - 5z + 6} \text{ invztrans, } z \rightarrow 4 \cdot 2^n \]
\[ y_{zs}(n) + y_{zi}(n) \rightarrow -\frac{35}{2} - 6 \cdot n + 28 \cdot 2^n - \frac{13}{2} \cdot 3^n \]

Do not forget to multiply the above Mathcad answers by the discrete unit-step function, \( u[k] \).
Suppose that we are only interested in the (coupled) complete response. Then, the solution is obtained as

\[ y(n) := H(z) \cdot F(z) + \frac{4z(z - 3)}{z^2 - 5z + 6} \]

\[ \xrightarrow{\text{invztrans, } z} \frac{-35}{2} - 6n + 28 \cdot 2^n - \frac{13}{2} \cdot 3^n \]

\[ k := 0 \ldots 4 \]

\[ y(k) = \begin{array}{c} 4 \\ 13 \\ 24 \\ 13 \\ -120 \end{array} \]
Next, we apply the Z-transform method to the advance formulation system,

\[ y[k + 2] - 5y[k + 1] + 6y[k] = f[k + 1] - 5f[k] \]

where, \( y[0] = 4 \), \( y[1] = 13 \) and \( f[k] = (3k + 5)u[k] \). Remember that the signals \( y[k + m] \) and \( f[k + m] \) in the advance formulation stand for \( y[k + m]u[k] \) and \( f[k + m]u[k] \), respectively.

Because, in this case, one of the initial conditions depends on the input to the system, we can only obtain the coupled solution as follows,

\[
\begin{align*}
 y[k + 2]u[k] &\leftrightarrow z^2Y(z) - z^2y[0] - zy[1] = z^2Y(z) - 4z^2 - 13z \\
 -5y[k + 1]u[k] &\leftrightarrow -5(zY(z) - zy[0]) = -5(zY(z) - 4z) \\
 6y[k]u[k] &\leftrightarrow 6Y(z)
\end{align*}
\]

\[
\begin{align*}
 f[k]u[k] = (3k + 5)u[k] &\leftrightarrow F(z) = \frac{z(5z - 2)}{(z - 1)^2} \\
 f[k + 1]u[k] &\leftrightarrow zF(z) - zf[0] = z \frac{z(5z - 2)}{(z - 1)^2} - 5z
\end{align*}
\]

Substituting the above expressions in the difference equation we get

\[
(z^2 - 5z + 6)Y(z) + (-4z^2 - 13z + 20z) = (z - 5) \frac{z(5z - 2)}{(z - 1)^2} - 5z
\]

Solve (Your turn) for \( Y(z) \) as one lumped rational function to obtain

\[
Y(z) = \frac{N(z)}{D(z)} = \frac{z(4z^3 - 15z^2 + z - 2)}{(z - 1)^2(z - 2)(z - 3)}
\]
which, after applying partial fraction expansion to \( \frac{Y(z)}{z} \) and then multiplying the result by \( z \), leads to

\[
Y(z) = -\frac{6z}{(z - 1)^2} - \frac{35}{2} \frac{z}{z - 1} + \frac{28z}{z - 2} - \frac{13}{2} \frac{z}{z - 3}
\]

Using the Z-transform Table leads to the following final result

\[
y[k] = Z^{-1}\{Y(z)\} = -6ku[k] - \frac{35}{2} u[k] + 28(2)^k u[k] - \frac{13}{2} 3^k u[k]
\]

By inspecting the above solution, we can identify the natural and forced responses (but not the zero-state and zero-input. Why?) as

\[
y_n[k] = 28(2)^k u[k] - \frac{13}{2} 3^k u[k]
\]

\[
y_f[k] = -\frac{35}{2} u[k] - 6ku[k]
\]

**Your turn:** Show that if you are to find \( y_{zs}[k] \) and \( y_{zi}[k] \) using the advance formulation for this system it would result in the following wrong decomposition (it does not agree with the decoupled delay formulation solution that we obtained earlier):

\[
y_{zi}[k] = 5(3)^k u[k] - (2)^k u[k]
\]

\[
y_{zs}[k] = -\frac{35}{2} u[k] - 6ku[k] - \frac{23}{2} 3^k u[k] + 29(2)^k u[k]
\]

However, the total (coupled) response obtained by adding these two signals would still be correct:

\[
y[k] = y_{zi}[k] + y_{zs}[k] =
\]

\[
= -6ku[k] - \frac{35}{2} u[k] + 28(2)^k u[k] - \frac{13}{2} 3^k u[k]
\]
**Example.** Consider the three-point moving average LTID system

\[ y[k] = \frac{1}{3}(f[k] + f[k - 1] + f[k - 2]) \]

and the discrete-time input signal shown below.

![Input Signal](image)

a. Find the impulse-response \( h[k] \). Is the system an FIR or IIR system?

b. Find the zero-state response \( y[k] \).

c. Verify and plot \( y[k] \) employing Mathcad.

**Solution:**

Applying the \( z \)-transform to the given recursion formula and using Pair 1 in the \( z \)-Transform Table we obtain (note that \( f[-1] = f[-2] = 0 \))

\[ Y(z) = \frac{1}{3} [F(z) + z^{-1}F(z) + z^{-2}F(z)] = \frac{1}{3} (1 + z^{-1} + z^{-2})F(z) \]

The transfer function is then

\[ H(z) = \frac{Y(z)}{F(z)} = \frac{1}{3} (1 + z^{-1} + z^{-2}) \]

The impulse response is obtained by applying the inverse \( z \)-transform to the above expression to obtain
\[ h[k] = \frac{1}{3} ([\delta] + \delta[k - 1] + \delta[k - 2]) \]

So, \( h[k] = \frac{1}{3} \) for \( k = 0, 1, 2 \) and zero otherwise. Therefore, the system has a finite impulse response (FIR).

The input is given by
\[ f[k] = \delta[k] + 2\delta[k - 1] - \delta[k - 2] + \delta[k - 3] + 2\delta[k - 4] \]
which has a \( z \)-transform given by
\[ F(z) = 1 + 2z^{-1} - z^{-2} + z^{-3} + 2z^{-4} \]

The zero-state response in the \( z \)-domain is
\[ Y(z) = H(z)F(z) = \frac{1}{3} (1 + z^{-1} + z^{-2})(1 + 2z^{-1} - z^{-2} + z^{-3} + 2z^{-4}) \]
\[ = \frac{1}{3} (1 + 3z^{-1} + 2z^{-2} + 2z^{-3} + 2z^{-4} + 3z^{-5} + 2z^{-6}) \]

Applying the inverse \( z \)-transform gives the zero-state response
\[ y[k] = \frac{1}{3} \delta[k] + \frac{2}{3} \delta[k - 1] + \frac{2}{3} \delta[k - 2] + \frac{2}{3} \delta[k - 3] + \frac{2}{3} \delta[k - 4] + \delta[k - 5] + \frac{2}{3} \delta[k - 6] \]

From the definition of the discrete-time unit-impulse function, we can express the response as the sequence (starting at \( k = 0 \))
\[ y[k] = \left\{ \frac{1}{3}, 1, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, 1, \frac{2}{3}, 0, 0, \ldots \right\} \]

**Your turn:** Show that the above sequence is identical to the one obtained using manual averaging of the input signal.
Mathcad verification session for the three-point moving average system:

\[
\text{Dirac}(k) := \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}
\]

\[
f(k) = \text{Dirac}(k) + 2\text{Dirac}(k - 1) - \text{Dirac}(k - 2) + \text{Dirac}(k - 3) + 2\text{Dirac}(k - 4)
\]

\[
F(z) = 1 + 2z^{-1} - z^{-2} + z^{-3} + 2z^{-4}
\]

\[
H(z) = \frac{1}{3}(1 + z^{-1} + z^{-2})
\]

\[
y(n) = H(z) \cdot F(z) \text{ invtrans}, z \rightarrow \frac{2}{3}\text{Dirac}(n - 6) + \frac{2}{3}\text{Dirac}(n - 5) + \frac{2}{3}\text{Dirac}(n - 4) + \frac{2}{3}\text{Dirac}(n - 3) + \frac{2}{3}\text{Dirac}(n - 2) + \text{Dirac}(n - 1) + \frac{1}{3}\text{Dirac}(n)
\]

\[n := -2..8\]

Since averaging is a form of low-pass filtering we see how the system supresses the sharp variations in the input signal.
**Example.** Determine the unit-impulse response for the following difference equation,

\[ y[k] - 2y[k - 1] + 2y[k - 2] = f[k] \]

We will first obtain the transfer function employing the formula

\[ H(z) = \frac{Y(z)}{F(z)} \mid_{i.c. = 0} \]

Z-transforming the equation (while setting initial conditions to zero) gives

\[ Y(z) - 2 \left( \frac{1}{z} Y(z) \right) + 2 \left( \frac{1}{z^2} Y(z) \right) = F(z) \]

\[ \left( \frac{2}{z^2} - \frac{2}{z} + 1 \right) Y(z) = F(z) \]

\[ H(z) = \frac{Y(z)}{F(z)} = \frac{1}{\frac{2}{z^2} - \frac{2}{z} + 1} = \frac{z^2}{z^2 - 2z + 2} \]

The poles are at \( z_{1,2} = 1 \pm j \). This system is unstable since \( |z_i| = \sqrt{2} > 1 \).

\( H(z) \) has complex conjugate poles. Therefore, we employ the Z-transform Pair 12c to find the unit-impulse response \( h[k] \),

\[ \frac{Az^2 + Bz}{z^2 + 2az + b^2} \leftrightarrow rb^k \cos(\beta k + \theta) u[k], \quad b > 0 \]

where,

\[ r = \sqrt{\frac{A^2b^2 + B^2 - 2AaB}{b^2 - a^2}}, \quad \beta = \cos^{-1}\left(\frac{-a}{b}\right) \]

\[ \theta = \tan^{-1}\left(\frac{Aa - B}{A\sqrt{b^2 - a^2}}\right) \]
By matching coefficients in

\[
\frac{Az^2 + Bz}{z^2 + 2az + b^2} = \frac{z^2}{z^2 - 2z + 2}
\]

we obtain,

\( A = 1, \ B = 0, b^2 = 2 \rightarrow b = +\sqrt{2} \) and \( a = -1 \). Then we use the above formulas to compute \( r, \beta \) and \( \theta \), as follows,

\[
\begin{align*}
    r &= \sqrt{2}, \quad \beta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4} \\
    \theta &= \tan^{-1}\left(\frac{-1 - 0}{1\sqrt{2} - 1}\right) = \tan^{-1}(-1) = -\frac{\pi}{4}
\end{align*}
\]

Therefore,

\[
\begin{align*}
    h[k] &= \sqrt{2}(\sqrt{2})^k \cos\left(\frac{\pi}{4} k - \frac{\pi}{4}\right) u[k] = (\sqrt{2})^{k+1}\cos\left(\frac{\pi}{4} k - \frac{\pi}{4}\right) u[k], \ k \geq 0
\end{align*}
\]

The system has an infinite impulse response (IIR). The first six output values are

\[ k := 0..5 \]
\[ h_k := (\sqrt{2})^{k+1}\cos\left(\frac{\pi}{4} k - \frac{\pi}{4}\right) \]
\[ h^T = (1 \ 2 \ 2 \ 0 \ -4 \ -8) \]

We can employ Simulink to solve the system (numerically) as shown below.
Alternative time-domain Simulink solution:

**Your turn:** Solve numerically for the first 10 terms of the sequence generated by the following recursion formula, \( u_{n+2} = 4u_{n+1} - 8u_n \). Assume that \( u_0 = u_1 = 1 \). Then, employ the Z-transform method to find the expression for the \( n \)th term, \( u_n \). Note: The above recursion formula can also be represented as the second-order homogeneous LTI difference equation \( y[n + 2] - 4y[n + 1] + 8y[n] = 0 \), with \( y[0] = y[1] = 1 \).

**Ans.** \( u_n = 2^{\frac{3n}{2}} \left[ 2\cos\left(\frac{nx}{4}\right) - \sin\left(\frac{nx}{4}\right) \right], n \geq 0 \)
The Amortization Problem. If you purchase a car, or a house, and take a loan of $d$ dollars, with fixed annual interest rate, $R$ (i.e., a rate of $r = R/12$ per month), then the loan is paid back through the process known in economics as amortization. Let us say you will be making fixed payments of $p$ dollars every month. What should your monthly payment be if you want to pay the entire loan within $N$ months? (Typically, $N = 48$ for a car and $N = 180, 240$ or 360 for a house).

Well, the outstanding principal at the end of a given month, $y[k + 1]$, is equal to the outstanding principal from the previous month, $y[k]$, plus the monthly interest accrued on that principal, $ry[k]$, minus the monthly payment, $f[k]$. We may express the above statement analytically as a recursive formula (difference equation):

$$y[k + 1] = y[k] + ry[k] - f[k]$$

which amounts to the difference equation

$$y[k + 1] - (1 + r)y[k] = -f[k]$$

With monthly payments (system input) $f[k] = pu[k]$ and initial principal, $y[0] = d$.

Applying the Z-transform to the difference equation we get

$$zY(z) - zy[0] - (1 + r)Y(z) = -F(z)$$

Solving for $Y(z)$ and substituting $F(z) = \frac{pz}{z-1}$ and $y[0] = d$, we obtain

$$Y(z) = \frac{-pz}{[z - (1 + r)](z - 1)} + \frac{dz}{z - (1 + r)}$$
Performing pfe on \( \frac{Y(z)}{z} \), we get

\[
Y(z) = \frac{-p}{r} \frac{1}{z - (1 + r)} + \left( \frac{p}{r} \right) \frac{1}{z - 1} + \frac{d}{z - (1 + r)}
\]

\[
= \left( d - \frac{p}{r} \right) \left( \frac{1}{z - (1 + r)} \right) + \left( \frac{p}{r} \right) \frac{1}{z - 1}
\]

Next, applying the inverse Z-transform

\[
y[k] = Z^{-1}\left\{ \left( d - \frac{p}{r} \right) \left( \frac{z}{z - (1 + r)} \right) + \left( \frac{p}{r} \right) \frac{z}{z - 1} \right\}
\]

leads to the desired solution for the owed principal after \( k \) months,

\[
y[k] = \left( \left( d - \frac{p}{r} \right) (1 + r)^k + \frac{p}{r} \right) u[k]
\]

If the loan is supposed to be paid back in full after \( N \) months, then we can solve for the required monthly payment by setting \( y[N] = 0 \) in the above equation to obtain,

\[
\left( \left( d - \frac{p}{r} \right) (1 + r)^N + \frac{p}{r} \right) = 0
\]

which has the following solution for the monthly payment

\[
p = \frac{r (1 + r)^N}{(1 + r)^N - 1} d
\]
For example, assume you get approved for a 30 year (360 months) $300,000 mortgage at an annual rate, $R = 4\%$. Then, the monthly payment would be

\[
p = \frac{\left(\frac{0.04}{12}\right) \left(1 + \frac{0.04}{12}\right)^{360}}{\left(1 + \frac{0.04}{12}\right)^{360} - 1} (300,000) = \$1,432.25
\]

The principal after 20 years is

\[
y[240] = \left(300,000 - \frac{1,432.25}{0.04/12}\right)(1 + 0.04/12)^{240} + \frac{1,432.25}{0.04/12} = \$141,461.67
\]

The \textit{total cost of the mortgage} is

\[
360p = 360(\$1432.25) = \$515,610
\]

which means that in this scenario your $300,000 house will cost you an additional $515,610 - $300,000 = $215,610 in interest!

Online calculators are available to solve the amortization problem. They use the same equations that were derived above. Here is one example:

![Mortgage calculator](image-url)
Your turn. National income is the total value a country’s output of all new goods and services produced in one year. The national income for year $k$, $y[k]$, is governed by the following (Samuelson model) set of difference equations [Paul Samuelson (1915-2009) a Nobel Prize-winning economist]:

$$ y[k] = c[k] + i[k] + f[k] $$

$$ c[k] = ay[k - 1] $$

$$ i[k] = b(c[k] - c[k - 1]) $$

where $a$ and $b$ are positive constants, $c[k]$ represents consumer expenditures, $i[k]$ is the induced private investment (new investment stimulated by an increase in demand), and $f[k]$ represents government expenditures.

Apply the Z-transform to the above system of equations and show that the transfer function is given by

$$ H(z) = \frac{Y(z)}{F(z)} = \frac{z^2}{z^2 - a(1 + b)z + ab} $$

Assume that the government makes a sustained expenditure of 1 (trillion dollars), $f[k] = u[k]$. Also, assume that the consumer’s “propensity to consume coefficient” is $a = 0.8$ (i.e., consumer expenditures are 80% of the national income). Solve for and plot $y_{zs}[k]$ for $b = 0, 0.25, 0.5$ and 0.9. Discuss the effects of the induced private investment “acceleration coefficient” $b$ on the national income. What will the national income be after many years? How does $b$ affect that value?
Numerical Integration: Trapezoidal Rule as a LTID System

Consider the integral (for convenience, we set the lower limit to zero)

\[ y_{true}(t) = \int_{0}^{t} f(\tau) d\tau \]

The trapezoidal rule for numerical integration leads to a discrete-time approximation \( y(t_k) \approx y_{true}(t)|_{t_k} \) according to the recursion

\[ y(t_{k+1}) = y(t_k) + \frac{h}{2} [f(t_{k+1}) + f(t_k)] \]

where \( k \) is an integer representing the iteration number, and \( y(0) = 0 \). The idea is to approximate the incremental area under the curve using a trapezoid of area \( \frac{h}{2} [f(t_{k+1}) + f(t_k)] \). The trapezoid’s base, \( h = t_{k+1} - t_k \), is a small positive constant (sampling period, \( h = T_s \)) defined by the user, as shown in the figure below.
An analysis based on Taylor series expansion leads to the total relative truncation error after \( n \) iterations, at time \( t_k = nh \), for (refer to Lecture 19 of Prof. Hassoun’s Numerical Methods course for detailed derivation)

We may compute an expression for the absolute relative error at \( t_k \) as,

\[
\left| \frac{y_{true}(t_k) - y(t_k)}{y_{true}(t_k)} \right| \approx \frac{1}{12} h^2 \left| \frac{f'(t_k) - f'(0)}{y_{true}(t_k)} \right|
\]

This result holds for any differentiable function \( f(t) \). The important thing to realize is that the trapezoidal rule has an error that is proportional to \( h^2 \); i.e., the error is \( O(h^2) \). It is interesting to note that if \( f(t) \) is linear (its derivative is a constant for all \( t \)) then \( f'(t) - f'(0) = 0 \). Thus, the trapezoidal rule is exact when integrating linear functions.

If we assume the sinusoid \( f(t) = \sin(\omega_o t) \), then the true value of the integral is

\[
y_{true}(t) = \int_0^t \sin(\omega_o \tau) d\tau = \frac{1}{\omega_o} (1 - \cos(\omega_o t))
\]

and it can be easily shown that

\[
\left| \frac{y_{true}(t_k) - y(t_k)}{y_{true}(t_k)} \right| \approx \frac{1}{12} h^2 \omega_o^2
\]

This last result makes qualitative sense: as \( \omega_o \) increases it causes the signal to have sharper variations and, therefore, the sampling interval, \( h \) must be reduced in order for the error not to increase. This result is consistent with sampling theory (refer to Lecture 13).
Next, we treat the trapezoidal rule as a difference equation and solve it for one specific function, \( f(t) = \sin(\omega_o t) u(t) \). In order to arrive at the difference equation, we transform the discrete-time signals \( y(t_k) \) and \( f(t_k) \) into the sequences \( y[k] \) and \( f[k] \).

The iterative version of the trapezoidal rule is given by,

\[
y[k + 1] = y[k] + \frac{h}{2} \{ f[k + 1] + f[k] \}
\]

Now, we are ready to solve for \( y[k] \). Employing the Z-transform we obtain

\[
z Y(z) - z y[0] - Y(z) = \frac{h}{2} \{ z F(z) - z f[0] + F(z) \}
\]

or, since we had assumed \( y[0] = y(0) = 0 \) and \( f[0] = \sin(0) = 0 \),

\[
(z - 1) Y(z) = \frac{h}{2} (z + 1) F(z)
\]

which leads to

\[
Y(z) = \frac{h}{2} \frac{z + 1}{z - 1} F(z)
\]

The transfer function of the discrete-time system is given by

\[
H(z) = \frac{Y(z)}{F(z)} = \frac{h}{2} \frac{z + 1}{z - 1}
\]

Letting \( f(t) = \sin(\omega_o t) u(t) \) leads to the sampled input sequence \( f[k] = \sin(\omega_o h k) u[hk] = \sin(\Omega_o k) u[k] \), with \( \Omega_o \equiv \omega_o h \). The Z-transform of \( f[k] \) is (using Pair 11b with \( \gamma = 1 \))

\[
F(z) = \frac{z \sin(\Omega_o)}{z^2 - 2z \cos(\Omega_o) + 1}
\]
Then,
\[ Y(z) = H(z)F(z) = \left(\frac{h}{2} z + 1\right) \frac{z \sin(\Omega_o)}{z^2 - 2z \cos(\Omega_o) + 1} \]

Performing partial fraction expansion on \( \frac{Y(z)}{z} \) we obtain (try it)
\[ \frac{Y(z)}{z} = \frac{h}{2} \left(\frac{-\sin(\Omega_o)}{\cos(\Omega_o) - 1}\right) \frac{1}{z - 1} + \frac{h}{2} \left(\frac{\sin(\Omega_o)}{\cos(\Omega_o) - 1}\right) \frac{z - \cos(\Omega_o)}{z^2 - 2z \cos(\Omega_o) + 1} \]

Finally, we employ the Z-transform Table to perform the inverse Z-transform of \( Y(z) \) (Pairs 2 and 11a), and obtain the solution
\[ y[k] = Z^{-1}\{Y(z)\} = \left(\frac{-h \sin(\Omega_o)}{2(\cos(\Omega_o) - 1)}\right) Z^{-1}\left\{ \frac{z}{z - 1} - \frac{z(z - \cos(\Omega_o))}{z^2 - 2z \cos(\Omega_o) + 1} \right\} \]
\[ y[k] = \frac{-h \sin(\Omega_o)}{2[\cos(\Omega_o) - 1]} [1 - \cos(\Omega_o k)] u[k] \quad (1) \]

By employing the earlier definition \( \Omega_o \triangleq \omega_o h \), we may write the (exact) sampled solution of the integral as
\[ y_{true}[k] = \frac{1}{\omega_o} (1 - \cos(\Omega_o k)) \quad (2) \]

For demonstration purposes, let us choose \( \omega_o = 10 \) and \( h = 0.08 \) (leading to \( \Omega_o = h \omega_o = 0.8 \)). The following plot compares the exact solution in (2) to the approximate one in (1), for \( t \in \left[0, \frac{\pi}{\Omega_o}\right] \). Here, \( \frac{\pi}{\Omega_o} \) represents half of the full period, \( 2\pi / \Omega_o \). The following Mathcad worksheet uses \( a \) for \( \Omega_o \).
We chose to plot the sampled exact solution as a continuous (red) curve for clarity. Obviously, if we reduce the value of $h$ we improve the approximation [recall that the error for the trapezoidal rule is proportional to $h^2$]. The following figure compares the results for $h = 0.02, 0.08$ and $0.2$ over two periods of the input.
Comparison between the Transfer Function of an Integrator and the Trapezoidal Rule-Based $H(z)$

We saw earlier that the trapezoidal integration rule can be viewed as a discrete-time LTI system with transfer function

$$H(z) = \frac{T z + 1}{2 z - 1} \tag{1}$$

The analog (exact) integrator transfer function is determined by applying the Laplace transform to the integration

$$y(t) = \int_0^t f(\tau) d\tau$$

which by means of the integration property of the Laplace transform gives

$$Y(s) = \frac{1}{s} F(s)$$

Therefore, the integrator transfer function is

$$H(s) = \frac{Y(s)}{F(s)} = \frac{1}{s} \tag{2}$$

Comparing (1) and (2) we conclude that we can arrive at $H(z)$ by using the mapping, $s = \frac{2 z - 1}{T z + 1}$. This is called the bilinear transformation and will be used in Lecture 24 in connection with digital filter design. We can also express the inverse of this transformation (your turn) as $z = \frac{2 + sT}{2 - sT}$ and show that the left half $s$-plane is mapped inside the unit circle in the $z$-plane.

Your Turn: Write a Matlab (or Mathcad) script that generates 1,000 random points in the left half $s$-plane (including the $j\omega$ axis) and maps and
plots these points in the $z$-plane. Assume $T = 1$ sec. According to your plot, where does the whole left half $s$-plane map into the $z$-plane?

We may also map a whole analog system to a discrete-time system using the bilinear transformation as demonstrated in the following example.

**Example.** Let $H(s) = \frac{2}{s+1}$ and $T = 0.1$. Then, we can arrive at the corresponding discrete-time transfer function using the substitution

$$s = \frac{2z - 1}{Tz + 1}$$

to get

$$H_1(z) = H(s) \big|_{s = \frac{2z - 1}{0.1z + 1}} = \frac{2}{20 \left(\frac{z-1}{z+1}\right) + 1} = \frac{2}{21} z + \frac{2}{21}$$

In Lecture 20, we employed the mapping $z = e^{sT}$ and arrived at the following discrete-time system transfer function approximation for the analog system $H(s) = \frac{2}{s+1}$ (in the limit of very small $T$ and the use of Taylor series approximation)

$$H_2(z) = \frac{2Tz}{z - e^{-T}} = \frac{2T}{1 - e^{-T}z^{-1}}$$

The following Mathcad simulation compares the step-response of the analog system, $H(s)$, and its two discrete-time system representations, $H_1(z)$ and $H_2(z)$. It is interesting to note that the bilinear approximation is more accurate than the Taylor expansion-based approximation.
\[ T := 0.1 \]

\[ f(t) := \Phi(t) \]

\[ H(s) := \frac{2}{s + 1} \]

\[ G_1(z) := H(s) \text{ substitute}, s = \frac{2 \cdot z - 1}{T \cdot z + 1} \rightarrow \left[ \frac{2}{20.0000000000000000000000} \cdot \frac{z - 1}{(z + 1)} + 1 \right] \]

\[ F_c(s) := f(t) \text{ laplace}, t \rightarrow \frac{1}{s} \quad F_d(z) := f(t) \text{ ztrans}, t \rightarrow \frac{z}{(z - 1)} \]

\[ y_c(t) := H(s) \cdot F_c(s) \text{ invlaplace}, s \rightarrow -2 \cdot \exp(-t) + 2 \]

\[ y_d1(n) := G_1(z) \cdot F_d(z) \text{ invztrans}, z \rightarrow 2.000 \cdot 1.000^n - 1.905 \cdot 0.9048^n \]

\[ y_d2(n) := \frac{2 \cdot T}{1 - e^{-T} \cdot z - 1} \cdot F_d(z) \text{ invztrans}, z \rightarrow 2.102 \cdot 1.000^n - 1.902 \cdot 0.9048^n \]

\[ n := 0 \ldots 100 \]
Your turn: Repeat the above simulations for the transfer function $H(s) = \frac{1}{s^2+1}$ with $T = 0.1$.

Your turn:

a. Show that Euler’s integration rule

$$y[k + 1] = y[k] + Tf[k + 1]$$

Leads to the mapping $z = \frac{1}{1-sT}$.

b. Write a Matlab (or Mathcad) script that generates 1,000 random points in the left half $s$-plane (including the $j\omega$ axis) and maps and plots these points in the $z$-plane according to $z = \frac{1}{1-sT}$. Assume $T = 1$ sec. According to your plot, where does the whole left half $s$-plane map into the $z$-plane?
Convolution Sums and LTI System Analysis

<table>
<thead>
<tr>
<th>No.</th>
<th>( f_1[k] )</th>
<th>( f_2[k] )</th>
<th>( f_1[k] \ast f_2[k] = f_2[k] \ast f_1[k] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \delta[k-j] )</td>
<td>( f[k] )</td>
<td>( f[k-j] )</td>
</tr>
<tr>
<td>2</td>
<td>( \gamma^k u[k] )</td>
<td>( u[k] )</td>
<td>( \left[ \frac{1 - \gamma^{k+1}}{1 - \gamma} \right] u[k] )</td>
</tr>
<tr>
<td>3</td>
<td>( u[k] )</td>
<td>( u[k] )</td>
<td>( (k+1)u[k] )</td>
</tr>
<tr>
<td>4</td>
<td>( \gamma^k u[k] )</td>
<td>( \gamma^k u[k] )</td>
<td>( \left[ \frac{\gamma_1^{k+1} - \gamma_2^{k+1}}{\gamma_1 - \gamma_2} \right] u[k] ) ( \gamma_1 \neq \gamma_2 )</td>
</tr>
<tr>
<td>5</td>
<td>( \gamma^k u[k] )</td>
<td>( \gamma_2 u[-(k+1)] )</td>
<td>( \frac{\gamma_1}{\gamma_2 - \gamma_1} \gamma_1^k u[k] + \frac{\gamma_2}{\gamma_2 - \gamma_1} \gamma_2^k u[-(k+1)] ) (</td>
</tr>
<tr>
<td>6</td>
<td>( k \gamma^k u[k] )</td>
<td>( \gamma_2^k u[k] )</td>
<td>( \frac{\gamma_1 \gamma_2}{(\gamma_1 - \gamma_2)^2} \left[ \gamma_2^k - \gamma_1^k + \frac{\gamma_1 - \gamma_2}{\gamma_2} k \gamma_1^k \right] u[k] ) ( \gamma_1 \neq \gamma_2 )</td>
</tr>
<tr>
<td>7</td>
<td>( ku[k] )</td>
<td>( ku[k] )</td>
<td>( \frac{1}{6} k(k-1)(k+1)u[k] )</td>
</tr>
<tr>
<td>8</td>
<td>( \gamma^k u[k] )</td>
<td>( \gamma^k u[k] )</td>
<td>( (k+1)\gamma^k u[k] )</td>
</tr>
<tr>
<td>9</td>
<td>( \gamma^k u[k] )</td>
<td>( ku[k] )</td>
<td>( \left[ \frac{\gamma^k(1-k) + k(1-\gamma)}{(1-\gamma)^2} \right] u[k] )</td>
</tr>
<tr>
<td>10</td>
<td>(</td>
<td>\gamma_1</td>
<td>^k \cos(\beta k + \theta) u[k] )</td>
</tr>
</tbody>
</table>

Note: For Pair 1 in the above table, \( j \) is a positive integer.

Just as we employed the convolution integral in the continuous-time domain to obtain the zero-state response of a LTI system, we can employ convolution sums to obtain the zero-state response of a discrete-time system, \( y_{zs}[k] = f[k] \ast h[k] \). The convolution sum generates a discrete-time signal according to the expression (refer to the section before last for derivation.)

\[
y_{zs}[k] = f[k] \ast h[k] = \sum_{n=-\infty}^{\infty} f[n] h[k - n]
\]
It should be emphasized that the convolution operation is *linear*, which means that the *commutative* property, *distributive* (over addition and subtraction) property, *associative* property, *scaling* property, *shift* property and *superposition* property apply. The following example illustrates the convolution analysis method.

**Example.** Employ convolution sums from the above Table to solve for the zero-state response of a LTID system to the input $f[k] = 2ku[k]$. Assume the system’s unit-impulse response is given by $h[k] = \delta[k] + \left(\frac{1}{2}\right)^k u[k]$.

**Solution.** In the following we employ the distributive property of convolution, Convolution Pair 1 (with $j = 0$) and Pair 6 (with $\gamma_1 = 1$)

$$y_{zs}[k] = f[k] \ast h[k] = 2ku[k] \ast \left[ \delta[k] + \left(\frac{1}{2}\right)^k u[k] \right]$$

$$= 2ku[k] \ast \delta[k] + 2ku[k] \ast \left(\frac{1}{2}\right)^k u[k]$$

$$= 2ku[k] + 2 \left(\frac{1}{2}\right) \left(\frac{1}{2}\right)^k - 1^k + \frac{1 - \frac{1}{2}}{\frac{1}{2}} k(1)^k \right) u(k)$$

$$= 2ku[k] + 4 \left(\frac{1}{2}\right)^k - 1 + k \right) u[k]$$

$$y_{zs}[k] = \left(4 \left(\frac{1}{2}\right)^k + 6k - 4\right) u[k]$$

Mathcad verification employing the $z$-transform method:
Example: Employ the definition of the convolution sum to solve for the zero-state sequence for the three-point moving average system.

Solution: The input signal and the impulse response are (as determined earlier) given by

\[ h[k] = \frac{1}{3}([\delta] + \delta[k - 1] + \delta[k - 2]) \]

\[ f[k] = \delta[k] + 2\delta[k - 1] - \delta[k - 2] + \delta[k - 3] + 2\delta[k - 4] \]

The zero-state response is given by

\[ y[k] = f[k] \ast h[k] = \sum_{n=-\infty}^{\infty} f[n]h[k-n] \]

Which can be written as (due to the causal and finite nature of the input),

\[ y[k] = \sum_{n=0}^{4} f[n]h[k-n] \]

For \( k = 0 \), we obtain

\[
y[0] = (1) \left( \frac{1}{3} \right) + (2)(0) + (-1)(0) + (1)(0) + (2)(0) = \frac{1}{3}
\]

For \( k = 1 \):
\[
\]
\[
= (1) \left( \frac{1}{3} \right) + (2) \left( \frac{1}{3} \right) + (-1)(0) + (1)(0) + (2)(0) = 1
\]

For \( k = 2 \):
\[
y[2] = \sum_{n=0}^{4} f[n]h[2 - n]
\]
\[
\]
\[
= (1) \left( \frac{1}{3} \right) + (2) \left( \frac{1}{3} \right) + (-1) \left( \frac{1}{3} \right) + (1)(0) + (2)(0) = \frac{2}{3}
\]

**Your turn:** Finish the above example by determining \( y[3] \), \( y[4] \) and \( y[5] \).

**Your turn:** Employ the Table of Convolution Sums to determine the zero-state response of the system \( h[k] = \left( \frac{1}{2} \right)^k u[k] \) to the following input \( f[k] \):

a. \( u[k] \)

b. \( \left( \frac{1}{3} \right)^k u[k] \)

c. \( 5 \left( \frac{1}{3} \right)^{k-2} u[k - 2] - 4 \left( \frac{1}{3} \right)^{k-4} u[k - 4] \)

Hint: employ the result from Part b, shift and superposition.
\[
\begin{aligned}
\{ & +1 \text{ if } k = 0 \\
& -1 \text{ if } k = 1 \\
& -1 \text{ if } k = 2 \\
& 0 \text{ otherwise}
\end{aligned}
\]

Hint: express the function as the sum of unit-steps and use the result form Part a, shift and superposition.

**Your turn:** Verify your solutions to the above problem employing the z-transform and Mathcad. Plot the responses as stem plots.
Derivation of the Convolution Sum for the Zero-State Response

We start by assuming a LTID system with unit-impulse response \( h[k] \). That is, for the input \( \delta[k] \) the system responds with \( h[k] \),

\[
\delta[k] \rightarrow \text{system} \rightarrow h[k]
\]

We are interested in showing that

\[
y_{zs}[k] = f[k] \ast h[k] = \sum_{n=-\infty}^{\infty} f[n]h[k - n]
\]

Let us express the signal \( f[k] \) as the sum

\[
f[k] = \cdots + f[-1]\delta[k + 1] + f[0]\delta[k] + f[1]\delta[k - 1] + \cdots
\]

\[
= \sum_{n=-\infty}^{\infty} f[n]\delta[k - n]
\]

Next, we employ the scaling and shift properties of linear systems and find the response to single sample input \( f[n]\delta[k - n] \) to be \( f[n]h[k - n] \).

\[
f[n]\delta[k - n] \rightarrow \text{system} \rightarrow f[n]h[k - n]
\]

Finally, applying the superposition property we obtain the zero-state response as the convolution sum

\[
y_{zs}[k] = \sum_{n=-\infty}^{\infty} f[n]h[k - n]
\]
Derivation of the Convolution Property of the Z-transform

In a similar fashion to the way we had derived the convolution property of the Laplace transform we shall derive the convolution property for the z-transform.

Assuming causal signals and Z-transforming the convolution sum that we have just derived, we obtain

\[ Y_{zs}(z) = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} f[n]h[k-n]z^{-k} \]

Interchanging the order of the summations in the above expression gives

\[ Y_{zs}(z) = \sum_{n=0}^{\infty} f[n] \sum_{k=0}^{\infty} h[k-n]z^{-k} \]

The right hand side sum is the z-transform of a right-shifted signal by \( n \) units, and can be represented as (employing the shift property of the z-transform)

\[ \sum_{k=0}^{\infty} h[k-n]z^{-k} = z^{-n}H(z) \]

Substituting the above sum in the expression for \( Y_{zs}(z) \) gives

\[ Y_{zs}(z) = \sum_{n=0}^{\infty} f[n]H(z)z^{-n} = H(z) \sum_{n=0}^{\infty} f[n]z^{-n} = H(z)F(z) \]

We have just proved that

\[ f[n] \ast h[n] \leftrightarrow F(z)H(z) \]