

ECE4330
Lecture 5
Signals (Cont.)
Prof. Mohamad Hassoun

Signal Average *Average over $t \in [a, b]$*

$$Ave_f \triangleq \frac{1}{b-a} \int_a^b f(t) dt$$

If $f(t)$ is periodic of period T_0 , then

$$Ave_f = \frac{1}{T_0} \int_0^{T_0} f(t) dt$$

Signal Energy

For real $f(t)$, the signal energy is defined as:

$$E_f \triangleq \int_{-\infty}^{+\infty} f^2(t) dt$$

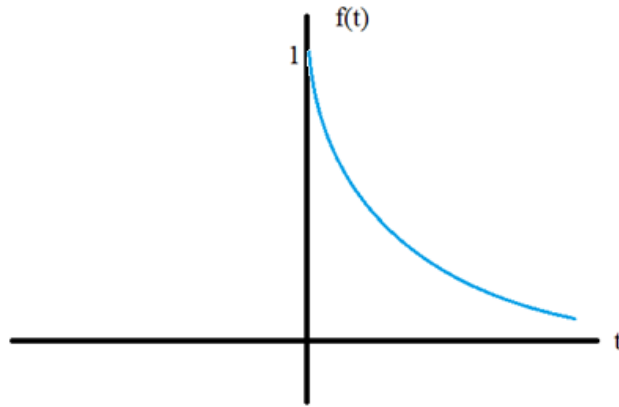
For a complex signal, the energy is given by

$$E_f \triangleq \int_{-\infty}^{+\infty} f(t) f^*(t) dt = \int_{-\infty}^{+\infty} |f(t)|^2 dt$$

(recall that if $z \in \mathbb{C}$, then $zz^* = z^*z = |z|^2$)

Example. Find the energy in the following signal.

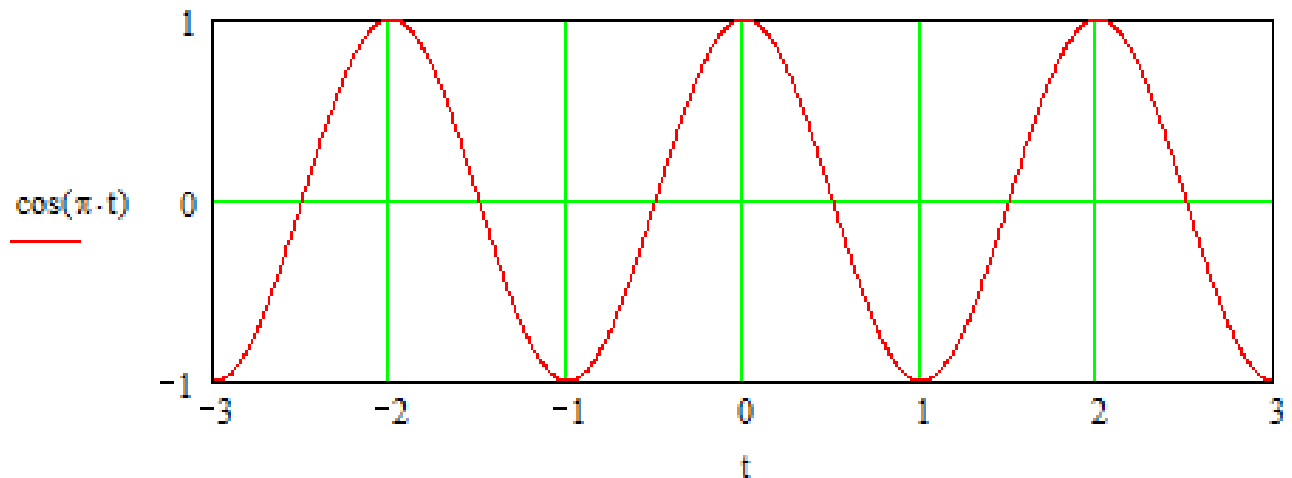
$$f(t) = \begin{cases} e^{-t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$



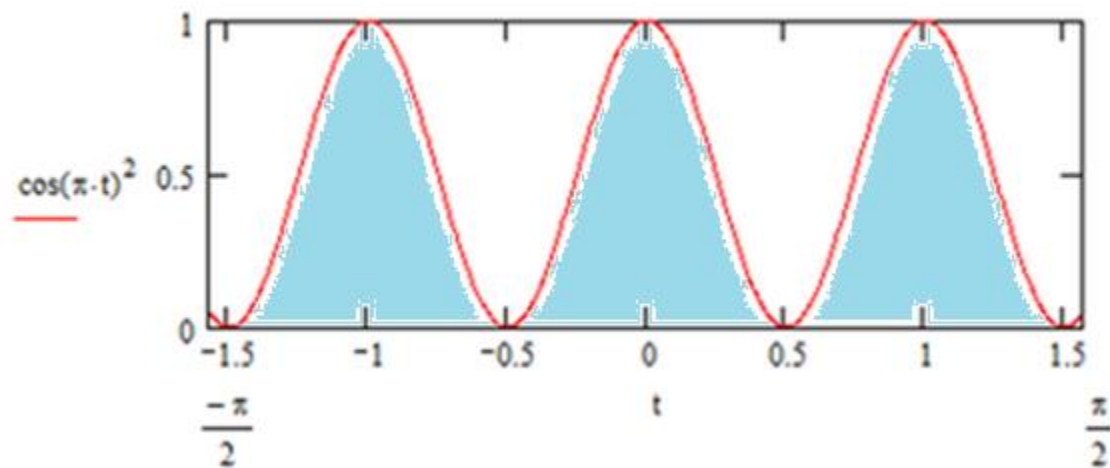
$$\begin{aligned} E_f &= \int_0^{\infty} (e^{-t})^2 dt = \int_0^{\infty} e^{-2t} dt = -\frac{1}{2} e^{-2t} \Big|_0^{\infty} \\ &= -\frac{1}{2} (e^{-\infty} - e^0) = \frac{1}{2} \text{ Joule} \end{aligned}$$

$f(t)$ is called an *energy signal* iff E_f is finite.

Example. $f(t) = \cos(\pi t)$, $\forall t$, is not an energy signal



$E_f = \int_{-\infty}^{+\infty} \cos^2(\pi t) dt = \infty$ (by inspection, the area under the curve of the squared function is infinite).



Analytically, we have

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\begin{aligned}
E_f &= \frac{1}{2} \int_{-\infty}^{+\infty} [1 + \cos(2\pi t)] dt = \frac{1}{2} \left[t + \frac{1}{2\pi} \sin(2\pi t) \right] \Big|_{-\infty}^{\infty} \\
&= \frac{1}{2} \lim_{\alpha \rightarrow \infty} \left(t + \frac{1}{2\pi} \sin(2\pi t) \right) \Big|_{-\alpha}^{\alpha} \\
&= \frac{1}{2} \lim_{\alpha \rightarrow \infty} \left(2\alpha + \frac{1}{\pi} \sin(2\pi\alpha) \right) = \infty + \frac{1}{2\pi} \sin(\infty) = \infty
\end{aligned}$$

$\sin(-x) = -\sin(x)$
 $\sin(\infty) \in [-1, +1]$

Therefore, $f(t)$ is not an energy signal because its energy is infinite.

Signal Average Power

$$P_f \triangleq \lim_{a \rightarrow \infty} \frac{1}{a} \int_{-\frac{a}{2}}^{\frac{a}{2}} |f(t)|^2 dt \quad \text{for real or complex } f(t)$$

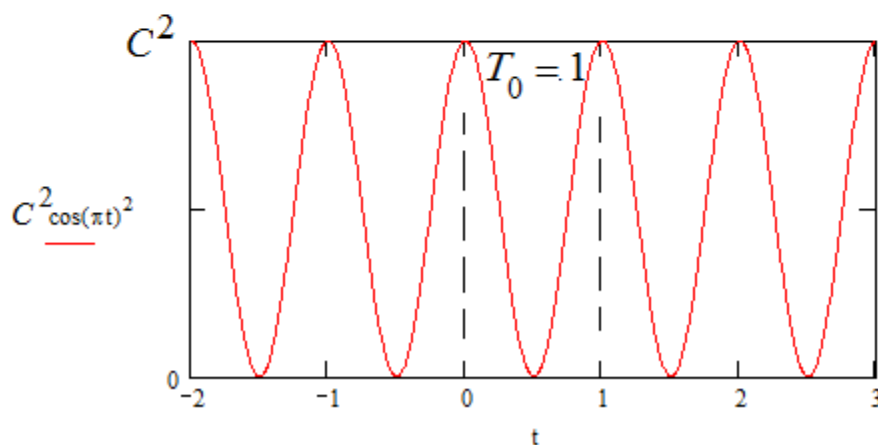
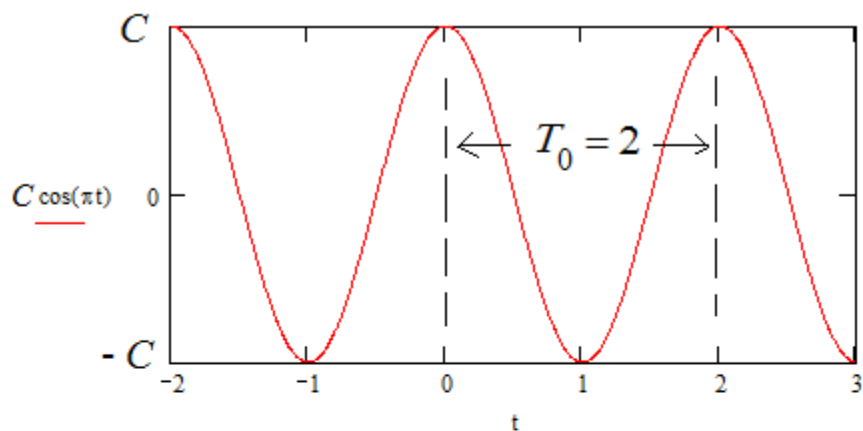
For the case of periodic, real $f(t)$ with period T :

$$P_f = \frac{1}{T} \int_T |f(t)|^2 dt = \frac{1}{T} \int_T f^2(t) dt$$

Note: Here, T can be the period of $f(t)$ or $f^2(t)$.

Example. Find the power in $f(t) = C \cos(\pi t)$.

$$T = \frac{2\pi}{\omega} = 2$$



Note: $T = 1$ for the squared signal. We may use $T = 2$ or 1.

$$\begin{aligned} P_f &= \frac{1}{1} \int_0^1 C^2 \cos^2(\pi t) dt = \frac{C^2}{2} \int_0^1 (1 + \cos(2\pi t)) dt \\ &= \frac{C^2}{2} \left[t + \frac{1}{2\pi} \sin(2\pi t) \right] \Big|_0^1 = \frac{C^2}{2} \end{aligned}$$

It can be shown that the above result is valid for any $f(t) = C\cos(\omega t + \theta)$, regardless of frequency and/or θ .

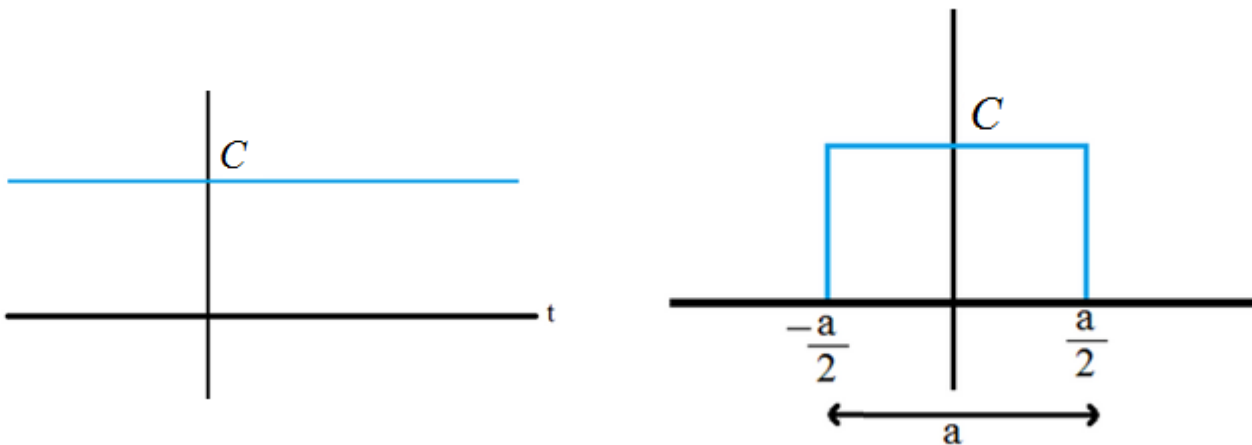
Your Turn: Prove the above fact using Mathcad.

Note:

$$\sqrt{P_f} = \sqrt{\frac{C^2}{2}} = \frac{C}{\sqrt{2}}$$

which is the root-mean-square (rms) or effective value for $f(t)$.

Power of a constant signal $f(t) = C$, is C^2 .



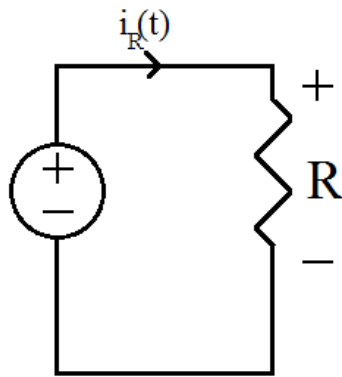
$$P_f = \lim_{a \rightarrow \infty} \frac{1}{a} \int_{-\frac{a}{2}}^{\frac{a}{2}} C^2 dt$$

$$= \lim_{a \rightarrow \infty} \frac{C^2}{a} t \bigg|_{-\frac{a}{2}}^{\frac{a}{2}} = \lim_{a \rightarrow \infty} \frac{C^2}{a} \left(\frac{a}{2} - \left(-\frac{a}{2} \right) \right) = \lim_{a \rightarrow \infty} \frac{aC^2}{a} = C^2$$

Circuit example (ac powered heater).

Let $V_o \cos(\omega t + \theta)$ be the voltage source signal in the following simple circuit. Then, the voltage across the resistor is:

$$v_R(t) = V_o \cos(\omega t + \theta)$$



Instantaneous power dissipated in the resistor:

$$\left(\text{Recall Ohm's Law: } v = Ri \rightarrow i = \frac{v}{R} \right)$$

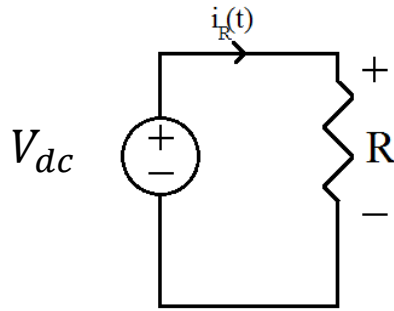
$$p_R(t) = v_R(t) \cdot i_R(t) = \frac{v_R^2(t)}{R}$$

$$\text{Average power: } P_R = \frac{1}{T_0} \int_{T_0} p_R(t) dt$$

$$P_R = \frac{1}{T_0} \int_{T_0} \frac{v_R^2(t)}{R} dt = \frac{1}{R} \left(\frac{1}{T_0} \int_{T_0} v_R^2(t) dt \right) = \frac{1}{R} \left(\frac{V_o^2}{2} \right) \text{ Watt}$$

(recall that the average power in a sinusoid of amplitude C is $C^2/2$)

Repeating the analysis for a dc source:



$$P_{dc} = \frac{V_{dc}^2}{R}$$

Let us equate the average power dissipated in the resistor due to each of the two sources:

$$P_{dc} = P_R \rightarrow \frac{V_{dc}^2}{R} = \frac{V_0^2}{2R}$$

$$V_{dc} = \sqrt{\frac{V_0^2}{2}} = \frac{V_0}{\sqrt{2}} \text{ Volt}$$

The above result shows that the heating effect due to an ac source is equivalent to that of a dc source that has the *rms*-value of the ac source. For example, the power outlet (in the USA) supplies a voltage that has amplitude $V_0 = 170$, then $V_{dc} = \frac{170}{\sqrt{2}} \cong 120$ Volts. In other words, a battery that has a voltage of 120 Volt will generate the same amount of heat as the ac outlet voltage. This result is independent of frequency and phase of the ac source. The same heater could work in Europe (since it is not affected by the frequency) and it would generate more power (effective voltage is 220) as long as the heating element can handle the increased current.

Power of the sum of two sinusoids:

$$f(t) = C_1 \cos(\omega_1 t + \theta_1) + C_2 \cos(\omega_2 t + \theta_2)$$

$$\begin{aligned} P_f &\triangleq \lim_{a \rightarrow \infty} \frac{1}{a} \int_{-\frac{a}{2}}^{\frac{a}{2}} f^2(t) dt && \xrightarrow{\quad} \frac{C_1^2}{2} && \xrightarrow{\quad} \frac{C_2^2}{2} \\ &= \lim_{a \rightarrow \infty} \left[\frac{1}{a} \int_{-\frac{a}{2}}^{\frac{a}{2}} C_1^2 \cos^2(\omega_1 t + \theta_1) dt + \frac{1}{a} \int_{-\frac{a}{2}}^{\frac{a}{2}} C_2^2 \cos^2(\omega_2 t + \theta_2) dt \right. \\ &\quad \left. + \frac{1}{a} \int_{-\frac{a}{2}}^{\frac{a}{2}} 2C_1 C_2 \cos(\omega_1 t + \theta_1) \cos(\omega_2 t + \theta_2) dt \right] \\ &&& \xrightarrow{\quad} ? \text{ Cross term} \end{aligned}$$

Case 1: $\omega_1 \neq \omega_2$

$$f(t) = C_1 \cos(\omega_1 t + \theta_1) + C_2 \cos(\omega_2 t + \theta_2)$$

Mathcad solution for average power:

$$f(t, C_1, C_2, \omega_1, \omega_2, \theta_1, \theta_2) := C_1 \cdot \cos(\omega_1 \cdot t + \theta_1) + C_2 \cdot \cos(\omega_2 \cdot t + \theta_2)$$

$$\lim_{a \rightarrow \infty} \frac{1}{a} \cdot \int_{-\frac{a}{2}}^{\frac{a}{2}} f(t, C_1, C_2, \omega_1, \omega_2, \theta_1, \theta_2)^2 dt \text{ simplify } \rightarrow \frac{1}{2} \cdot C_2^2 + \frac{1}{2} \cdot C_1^2$$

No contribution from the cross term. i.e., The average power is a constant, regardless of the phases θ_1 and θ_2 .

Animation for the case: $\omega_1 \neq \omega_2$

Case 2 $\omega_1 = \omega_2 = \omega$

$$f(t) = C_1 \cos(\omega t + \theta_1) + C_2 \cos(\omega t + \theta_2) \quad (\text{Equal frequencies})$$

Mathcad solution for average power:

$$\lim_{a \rightarrow \infty} \frac{1}{a} \cdot \int_{-\frac{a}{2}}^{\frac{a}{2}} f(t, C_1, C_2, \omega, \theta_1, \theta_2)^2 dt \text{ simplify} \rightarrow \frac{1}{2} \cdot C_1^2 + C_1 \cdot C_2 \cdot \cos(\theta_1) \cdot \cos(\theta_2) + C_1 \cdot C_2 \cdot \sin(\theta_1) \cdot \sin(\theta_2) + \frac{1}{2} \cdot C_2^2$$

or

$$\frac{\omega}{2\pi} \cdot \int_0^{2\pi} (C_1 \cdot \cos(\omega \cdot t + \theta_1) + C_2 \cdot \cos(\omega \cdot t + \theta_2))^2 dt \text{ simplify} \rightarrow \frac{1}{2} \cdot C_1^2 + C_1 \cdot C_2 \cdot \cos(\theta_1) \cdot \cos(\theta_2) + C_1 \cdot C_2 \cdot \sin(\theta_1) \cdot \sin(\theta_2) + \frac{1}{2} \cdot C_2^2$$

$$P_f = \frac{C_1^2}{2} + \frac{C_2^2}{2} + C_1 C_2 [\cos(\theta_1) \cos(\theta_2) + \sin(\theta_1) \sin(\theta_2)]$$

Employing the proper trig identity, we obtain:

$$P_f = P_f(\theta_1, \theta_2) = \frac{C_1^2}{2} + \frac{C_2^2}{2} + C_1 C_2 \cos(\theta_1 - \theta_2)$$

(Refer to Problem 7 in Assignment 2 for an alternative derivation)

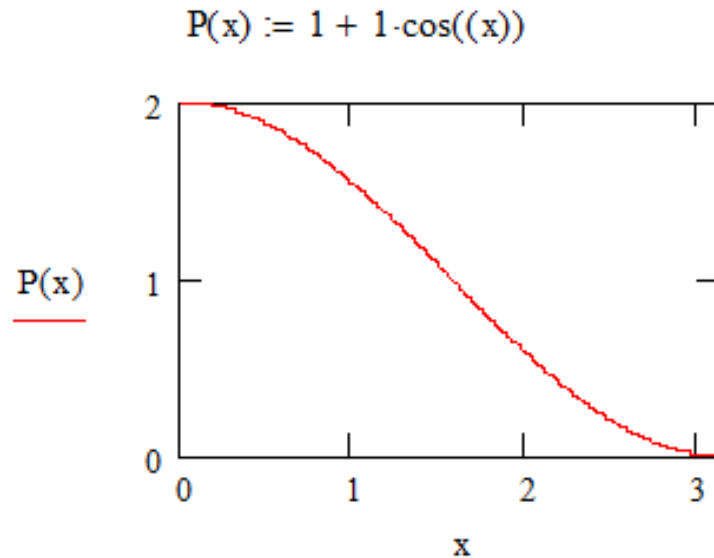
Some special cases:

Case i: $\theta_1 = \theta_2 \rightarrow P_f = \frac{C_1^2}{2} + \frac{C_2^2}{2} + C_1 C_2$

Case ii: $\theta_1 - \theta_2 = \pi \rightarrow P_f = \frac{C_1^2}{2} + \frac{C_2^2}{2} - C_1 C_2$

Case iii: $\theta_1 - \theta_2 = \frac{\pi}{2} \rightarrow P_f = \frac{C_1^2}{2} + \frac{C_2^2}{2}$

Note that if $\omega_1 = \omega_2$ and $C_1 = C_2 = 1$ and $x = \theta_1 - \theta_2$ then the average power can be plotted as shown, $P(x) = 1 + \cos(x)$,



According to the plot, the power is maximized when the sinusoids have the same phase. The power is minimized when the sinusoids are out of phase.

[Animation for the case:](#) $\omega_1 = \omega_2$

The above result for the case $\omega_1 \neq \omega_2 \neq \omega_3 \neq \dots$ can be generalized to the signal (with average C_0),

$$f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(\omega_n t + \theta_n),$$

to obtain,

$$P_f = C_0^2 + \frac{C_1^2}{2} + \frac{C_2^2}{2} + \dots = C_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} C_n^2$$

Paresevall's Theorem: Consider a real *periodic* signal $f(t)$, then the average power of the signal can be expressed as

$$\frac{1}{T} \int_T f^2(t) dt = C_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} C_n^2$$

where C_i are the coefficients in the following Fourier series expansion:

$$f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(\omega_n t + \theta_n)$$

Complex Exponential Signals:

$$f(t) = D e^{j\omega t}$$

D is a complex constant

$$P_f = \lim_{a \rightarrow \infty} \frac{1}{a} \int_{-\frac{a}{2}}^{\frac{a}{2}} |f|^2 dt = \lim_{a \rightarrow \infty} \frac{1}{a} \int_{-\frac{a}{2}}^{\frac{a}{2}} f f^* dt$$

Using the complex number identities $(z_1 z_2)^* = z_1^* z_2^*$ and $z z^* = |z|^2$, we may write

$$\begin{aligned} f \cdot f^* &= D e^{j\omega t} \cdot (D e^{j\omega t})^* = D e^{j\omega t} \cdot D^* e^{-j\omega t} \\ &= D D^* e^{(j\omega t - j\omega t)} = D D^* = |D|^2 \end{aligned}$$

$$P_f = \lim_{a \rightarrow \infty} \frac{1}{a} \int_{-\frac{a}{2}}^{\frac{a}{2}} |D|^2 dt = \lim_{a \rightarrow \infty} \frac{|D|^2}{a} t \bigg|_{-\frac{a}{2}}^{\frac{a}{2}} = |D|^2$$

Power of the sum of two complex exponential signals:

$$f(t) = D_1 e^{j\omega_1 t} + D_2 e^{j\omega_2 t}$$

$$\text{Case 1: } \omega_1 \neq \omega_2 \rightarrow P_f = |D_1|^2 + |D_2|^2$$

$$\text{Case 2: } \omega_1 = \omega_2 \rightarrow P_f = |D_1|^2 + |D_2|^2 + (D_1^* D_2 + D_1 D_2^*)$$

[Your turn: Verify these results using Mathcad integration]

The following is a derivation of Case 2 ($\omega_1 = \omega_2 = \omega$):

$$\begin{aligned} f(t)f(t)^* &= |f(t)|^2 = (D_1 e^{j\omega t} + D_2 e^{j\omega t})(D_1 e^{j\omega t} + D_2 e^{j\omega t})^* \\ &= D_1 D_1^* e^{j\omega t} e^{-j\omega t} + D_1 D_2^* e^{j\omega t} e^{-j\omega t} + D_2 D_1^* e^{j\omega t} e^{-j\omega t} \\ &\quad + D_2 D_2^* e^{j\omega t} e^{-j\omega t} \end{aligned}$$

We used the fact that the complex conjugate of a sum is the sum of the conjugates. Realizing that $e^{j\omega t} e^{-j\omega t} = e^{j\omega t - j\omega t} = e^0 = 1$, we can simplify the above expression to get a constant value,

$$\begin{aligned} |f(t)|^2 &= D_1 D_1^* + D_1 D_2^* + D_2 D_1^* + D_2 D_2^* \\ &= |D_1|^2 + |D_2|^2 + D_1^* D_2 + D_1 D_2^* \end{aligned}$$

Therefore, the power of $f(t)$ is then the *average* of a constant, which is the constant itself

$$P_f = \lim_{a \rightarrow \infty} \frac{1}{a} \int_{-\frac{a}{2}}^{\frac{a}{2}} |f(t)|^2 dt = |D_1|^2 + |D_2|^2 + D_1^* D_2 + D_1 D_2^*$$

Example. Find the average power in $f(t) = 2e^{j(5t + \frac{\pi}{4})}$

$$D = 2e^{j(\frac{\pi}{4})}, |D| = 2$$

$$P_f = |D|^2 = 2^2 = 4$$

Example. Find the average power in

$$f(t) = (1 + j)e^{j5\pi t} + (j)e^{j3\pi t}$$

$$D_1 = 1 + j \text{ and } D_2 = j$$

$$\omega_1 \neq \omega_2 \rightarrow P_f = |1 + j|^2 + |j|^2 = 2 + 1 = 3$$

Example. Find the average power for the following signal,

$$f(t) = \underbrace{(1 + j)}_{D_1} e^{j3\pi t} + \underbrace{(j)}_{D_2} e^{j3\pi t}$$

$$\begin{aligned} \omega_1 = \omega_2 \rightarrow P_f &= |1 + j|^2 + |j|^2 + (1 + j)^*(j) + (1 + j)(j)^* \\ &= 2 + 1 + (1 - j)(j) + (1 + j)(-j) \\ &= 2 + 1 + j + \cancel{1} - j + \cancel{1} \\ &= 5 \end{aligned}$$

Your turn: Show that the magnitude of the signal ($a \neq b$)

$$f(t) = e^{jat} + e^{jbt}$$

$$\text{is } |f(t)| = 2 \left| \cos \left(\frac{a-b}{2} t \right) \right|.$$

- Assume, $a = 2$ and $b = 3$. Plot $|f(t)|$ for $t \in [-\pi \ 3\pi]$.
- What is the period of $|f(t)|$?
- Determine the average power of $f(t)$ using the formula,
 $P_f = |D_1|^2 + |D_2|^2$.
- Repeat employing the definition of average power,

$$P_f = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)f(t)^* dt = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |f(t)|^2 dt$$

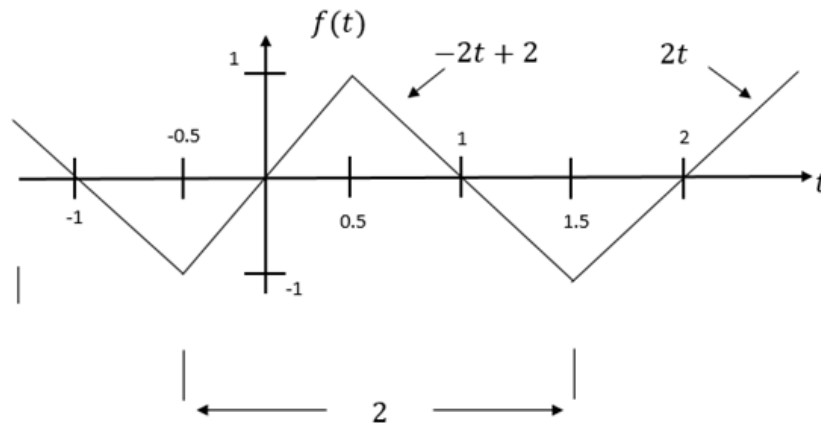
Hint: you can save a lot of trigonometry in finding $|f(t)|$ if you employ Euler's identity after expressing $f(t)$ as

$$f(t) = e^{jat} + e^{jbt} = \left(e^{j\frac{a-b}{2}t} + e^{-j\frac{a-b}{2}t} \right) e^{j\frac{a+b}{2}t}$$

Example. Find the average, energy and the average power for the following signal (with period $T_0 = 2$):

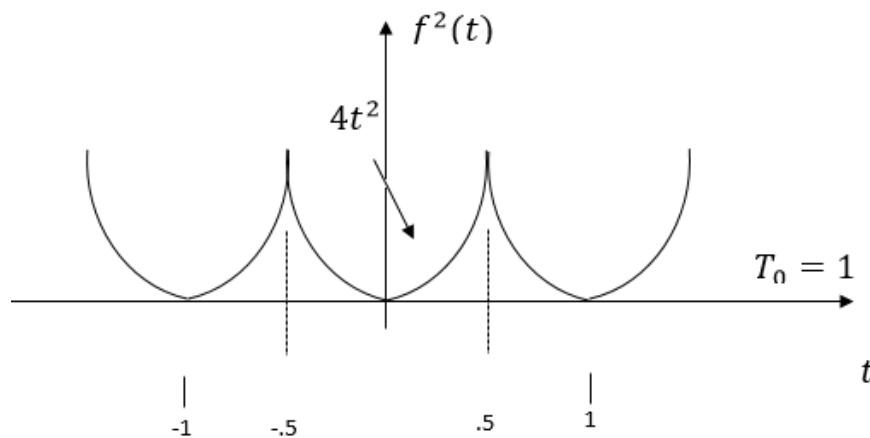
$$f(t) = \begin{cases} 2t & -\frac{1}{2} \leq t \leq \frac{1}{2} \\ -2t + 2 & \frac{1}{2} \leq t \leq \frac{3}{2} \end{cases}$$

The following is a plot of $f(t)$.



$Ave_f = 0$ (by inspection of the area under curve of f)

$E_f = \infty$ (by inspection of area under the curve of f^2)



$$P_f = \frac{1}{2} \int_{T_0} f^2(t) dt = \frac{1}{2} \left[\int_{-\frac{1}{2}}^{\frac{1}{2}} (2t)^2 dt + \int_{\frac{1}{2}}^{\frac{3}{2}} (2 - 2t)^2 dt \right] = \frac{1}{3}$$

Alternative solution for average power computation: Note that the period of the squared signal is one. The average power is then

$$P_f = \frac{1}{1} \int_{-\frac{1}{2}}^{\frac{1}{2}} 4t^2 dt = \frac{4}{3} t^3 \bigg|_{-\frac{1}{2}}^{\frac{1}{2}} = \frac{4}{3} \left[\left(\frac{1}{2}\right)^3 - \left(-\frac{1}{2}\right)^3 \right] = \frac{1}{3}$$

The Fourier series for the above triangular signal $f(t)$ is given by (refer to your textbook, pp. 197-198):

$$f(t) = \frac{8}{\pi^2} \left[\cos\left(\pi t - \frac{\pi}{2}\right) + \frac{1}{9} \cos\left(3\pi t + \frac{\pi}{2}\right) + \frac{1}{25} \cos\left(5\pi t - \frac{\pi}{2}\right) + \frac{1}{49} \cos\left(7\pi t + \frac{\pi}{2}\right) + \dots \right]$$

Since $\omega_1 \neq \omega_2 \neq \dots \neq \omega_n$, the average signal power is given by

$$P_f = \sum_{n=1}^{\infty} \frac{C_n^2}{2} = \left(\frac{8^2}{2\pi^4} \right) \left[1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \dots \right]$$

By Parseval's Theorem, we write

$$P_f = \frac{1}{3} = \left(\frac{8^2}{2\pi^4} \right) \left[1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \dots \right]$$

Upon dividing the above expression by $\frac{8^2}{2\pi^4}$ we obtain

$$1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \dots = \frac{\pi^4}{96}$$

We have just computed the exact sum of a series! This is one of the explicit series for π that we encountered in Lecture 3.

Your Turn: Consider a real-valued signal written in terms of its even and odd components, $f(t) = f_e(t) + f_o(t)$. Show that:

- a. $\int_{-\infty}^{\infty} f_o(t) dt = 0$.
- b. The product $f_o(t)f_e(t)$ is an odd function.
- c. The average power of $f(t)$ is the sum of the average powers of $f_e(t)$ and $f_o(t)$.