Linear Time-Invarient Systems

$f \longrightarrow$ **(LTI)** $\longrightarrow y = y_{zi} + y_{zs}$

$f(t) \longleftrightarrow f[k]$

Continuous-Time
(*linear differential equation*)

Laplace Transform

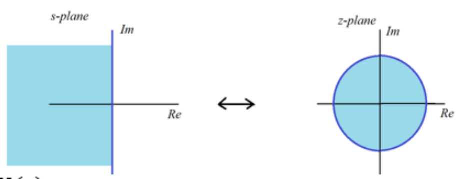
Use to find the complete response $y(t)$
for stable/unstable LTI system

$$F(s) = \int_{0^-}^{\infty} f(t)e^{-st}\,dt$$

Transfer function: $H(s) = \frac{Y(s)}{F(s)}$

Stability requires $Re(s_i) < 0$ for all poles s_i of $H(s)$.

Sampling Theorem: Sampling rate $f_s = \frac{1}{T_s} > 2B$
(B is the bandwidth of $f(t)$)



Discrete-Time
(*linear difference equation*)

Z-Transform

Use to find the complete response $y[z]$
for stable/unstable LTI system

$$F(z) = \sum_{k=-\infty}^{\infty} f[k]z^{-k}$$

Transfer function: $H(z) = \frac{Y(z)}{F(z)}$

Stability requires $|\gamma_i| < 1$, for all poles γ_i of $H(z)$

Fourier Transform

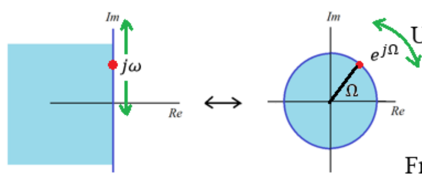
Useful for the analysis of stable LTICT systems

$$F(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt$$

Frequency response function $H(\omega) = H(s)|_{s=j\omega}$

If $f(t)$ is everlasting: $y_{ss}(t) = F^{-1}\{H(\omega)F(\omega)\}$

Applications: Signal bandwidth, analog filter design, sampling and communication systems, etc.



Discrete-Time Fourier Transform

Useful for the analysis of stable LTIDT systems

$$F(\Omega) = \sum_{k=-\infty}^{\infty} f[k]e^{-j\Omega k}$$

Frequency response function: $H(\Omega) = H(z)|_{z=e^{j\Omega}}$

If $f[k]$ is everlasting: $y_{ss}[k] = F^{-1}\{H(\Omega)F(\Omega)\}$

Applications: Recursive (digital) filter design

Linear Time-Invariant (LTI) Systems

- Continuous-Time (CT) systems process CT signals.
- Discrete-Time (DT) systems process DT signals.
- $f(t)$ represents a continuous-time signal.
- $f[k]$ represents a discrete-time signal.
- Bandwidth of $f(t)$ is B (Hz). B is determined by applying the Fourier transform.
- Sampled signal: $f(t) \rightarrow f[k] = f(T_s k)$, with integer k.
- Sampling Theorem: $T_s < \frac{1}{2B}$ is required for perfect reconstruction of $f(t)$ from $f[k]$.
- LTICT system dynamics are governed by a *linear differential equation* which relates the system response $y(t)$ to the input signal $f(t)$, subject to certain initial conditions.
- LTIDT system dynamics are governed by a *linear difference equation* which relates the system response $y[k]$ to the input signal $f[k]$, subject to certain initial conditions.
- System analysis methods allow us to solve for $y = y_{zi} + y_{zs}$, where y_{zi} is the (zero-input) response due to initial conditions (or energy stored in the system) and y_{zs} is the (zero-state) response due to the input f (with stored energy set to zero).
- System Transfer functions are *rational functions* (ratio of two polynomials) that are obtained using the Laplace transform for LTICT systems, and using the Z-transform for LTIDT systems.

- LTICT system transfer function: $H(s) = \frac{N(s)}{D(s)}$ [$D(s)$ is the characteristic equation of the system].
- The solutions, s_i, of $D(s) = 0$ are the system poles or natural frequencies (generally complex)
- LTICT system natural modes take the form, $e^{s_i t}$. Stability requires $Re(s_i) < 0$ for all s_i.
- LTIDT system transfer function: $H(z) = \frac{N(z)}{D(z)}$ [$D(z)$ is the characteristic equation of the system]
- The solutions, γ_i, of $D(z) = 0$ are the system poles or natural frequencies (generally complex).
- LTIDT system natural modes take the form, γ_i^k. Stability requires $|\gamma_i| < 1$ for all γ_i.

Fourier Series:

- The Fourier series allows us to express a periodic signal $f(t)$, with a fundamental frequency ω_o, as the sum of a constant (signal average) and an infinite number of harmonic sinusoids:
- Compact trigonometric Fourier series: $f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_o t + \theta_n)$
- Exponential Fourier series: $f(t) = \sum_{n=-\infty}^{+\infty} D_n e^{jn\omega_0 t}$
- Parseval's theorem: $\frac{1}{T_0}\int_{T_0} f^2(t)dt = C_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} C_n^2 = D_0^2 + 2\sum_{n=1}^{+\infty}|D_n|^2$

Laplace Transform:

$$F(s) = \int_{0^-}^{\infty} f(t)e^{-st}\,dt$$

- Transforms a CT signal, $f(t)$, into a complex signal, $F(s)$, in the s-domain ($s = \sigma + j\omega$).
- Table of Laplace transform pairs $f(t) \leftrightarrow F(s)$ and Laplace transform properties are available that allow us to find $F(s)$ from $f(t)$ and $f(t)$ from $F(s)$, and avoid using the above integral.
- The Laplace transform simplifies the analysis of a LTICT system by transforming its linear differential equation into an algebraic equation: $Q(s)Y(s) + I(s) = P(s)F(s)$, where $y(t) \leftrightarrow Y(s)$ and $f(t) \leftrightarrow F(s)$. Here, $Q(s)$, $P(s)$ and $I(s)$ are polynomials. $I(s)$ depends only on the system's initial conditions.
- Such algebraic equation can then be solved to obtained the transfer function $H(s) = \frac{Y(s)}{F(s)} = \frac{P(s)}{Q(s)}$, where $I(s)$ is set to zero (i.e., initial conditions are set to zero).
- The system's unit-impulse response, $h(t)$, and the transfer function, $H(s)$, form a Laplace transform pair: $h(t) \leftrightarrow H(s)$
- The Laplace transform can be used to analyze stable and unstable LTICT systems. It can generate the response: $Y(s) = Y_{zi}(s) + Y_{zs}(s)$, where $Y_{zi}(s) = -\frac{I(s)}{Q(s)}$ and $Y_{zs}(s) = \frac{P(s)}{Q(s)}F(s) = H(s)F(s)$
- Performing the inverse Laplace transformation on $Y_{zi}(s) = -\frac{I(s)}{Q(s)}$ and $Y_{zs}(s) = H(s)F(s)$ (which involves applying partial fraction expansion and the use of the Laplace transform Table) lead to the desired time-domain solution:

$$y(t) = y_{zi}(t) + y_{zs}(t) = L^{-1}\left\{-\frac{I(s)}{Q(s)}\right\} + L^{-1}\{H(s)F(s)\}$$

- In addition to the analysis of LTICT systems, the Laplace transform finds important applications in automatic control.

Fourier Transform:

$$F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt$$

- Transforms a CT signal, $f(t)$, into a complex signal, $F(\omega)$, in the ω-domain.
- Tables of Fourier transform pairs, $f(t) \leftrightarrow F(\omega)$, and Fourier transform properties are available to assist in obtaining the transform and its inverse.
- If a causal signal $f(t)$ is absolutely integrable (i.e., $\int_{0^-}^{\infty} |f(t)| dt < \infty$), and has a Laplace transform $F(s)$, then (and only then) the Fourier transform of $f(t)$ can be obtained from $F(s)$ as
$$F(\omega) = F(s)|_{s=j\omega}$$
- Similarly, the frequency response function $H(\omega)$ of a stable LTICT system can be obtained from the transfer function $H(s)$ by simply replacing s by $j\omega$.
- The Fourier transform cannot handle initial conditions like the Laplace transform. Therefore, it cannot be used to obtain the $y_{zi}(t)$ component of the response. However, if the system is stable, the Fourier transform can be used to obtain the zero-state response according to the formula: $y_{zs}(t) = F^{-1}\{H(\omega)F(\omega)\}$. Here, the notation $F^{-1}\{\}$ stands for the inverse Fourier transform.
- If $f(t)$ is a constant, an everlasting sinusoid or (more generally) an everlasting periodic signal then $F^{-1}\{H(\omega)F(\omega)\}$ gives the steady-state response, $y_{ss}(t)$.
- The response of a stable system to a periodic $f(t)$, that is expressed in terms of its trigonometric Fourier series, $f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_o t + \theta_n)$, is given by the formula
$$y_{ss}(t) = H(0)C_0 + \sum_{n=1}^{\infty} |H(n\omega_0)| C_n \cos[n\omega_0 t + \angle H(n\omega_0) + \theta_n]$$
- For a periodic $f(t)$ that is expressed in terms of its exponential Fourier series $f(t) = \sum_{n=-\infty}^{+\infty} D_n e^{jn\omega_0 t}$ the steady-state response is determined using the formula,
$$y_{ss}(t) = \sum_{n=-\infty}^{+\infty} H(n\omega_0) D_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{+\infty} |H(n\omega_0)||D_n| e^{j[n\omega_0 t + \angle H(n\omega_0) + \angle D_n]}$$
- The Fourier transform is very useful in many applications including the analysis and design of communication systems, design of analog filters, sampling theory, and in determining a signal's bandwidth. We can express the *bandwidth equation* for a signal $f(t)$ as (α is typically set to a value between 0.95 and 0.99).
$$\frac{\int_0^{\omega_B} |F(\omega)|^2 d\omega}{\int_0^{\infty} |F(\omega)|^2 d\omega} = \alpha$$
- The above equation can be solved (analytically or numerically) for ω_B from which we define the *signal's effective bandwidth* (in Hz) as
$$B = \frac{\omega_B}{2\pi} \text{ Hz}$$

The Z-Transform:

$$F(z) = \sum_{k=-\infty}^{\infty} f[k]z^{-k}$$

- Transforms a DT signal, $f[k]$, into a complex signal, $F(z)$, in the z-domain (z is complex)
- Tables of Z-transform pairs, $f[k] \leftrightarrow F(z)$, and Z-transform properties are available to assist in obtaining the transform and its inverse.
- The Z-transform simplifies the analysis of a LTIDT system by transforming its linear difference equation into an algebraic equation: $Q(z)Y(z) + I(z) = P(z)F(z)$, where $y[k] \leftrightarrow Y(z)$ and $f[k] \leftrightarrow F(z)$. Here, $Q(z)$, $P(z)$ and $I(z)$ are polynomials. $I(z)$ depends only on the system's initial conditions.
- Such algebraic equation can then be solved to obtained the transfer function $H(z) = \frac{Y(z)}{F(z)} = \frac{P(z)}{Q(z)}$, where $I(z)$ is set to zero (i.e., initial conditions are set to zero).
- The system's unit-impulse response, $h[k]$, and the transfer function, $H(z)$, form a Z-transform pair: $h[k] \leftrightarrow H(z)$
- The Z-transform can be used to analyze stable and unstable LTIDT systems. It can generate the response: $Y(z) = Y_{zi}(z) + Y_{zs}(z)$, where $Y_{zi}(z) = -\frac{I(z)}{Q(z)}$ and $Y_{zs}(z) = \frac{P(z)}{Q(z)}F(z) = H(z)F(z)$
- Performing the inverse Z-transformation on $Y_{zi}(z) = -\frac{I(z)}{Q(z)}$ and $Y_{zs}(z) = H(z)F(z)$ (which involves applying partial fraction expansion and the use of the Z-transform Table) lead to the desired time-domain solution:

$$y[k] = y_{zi}[k] + y_{zs}[k] = Z^{-1}\left\{-\frac{I(z)}{Q(z)}\right\} + Z^{-1}\{H(z)F(z)\}$$

Discrete-Time Fourier Transform (DTFT):

$$F(\Omega) = \sum_{k=-\infty}^{\infty} f[k]e^{-j\Omega k}$$

where $\Omega = T_s\omega = \frac{\omega}{f_s}$ is the *discrete-time angular frequency* (measured in rad/sample), and f_s is the sampling frequency.

- The DTFT transforms a DT signal, $f[k]$, into a complex signal, $F(\Omega)$, in the Ω-domain.
- Tables of DTFT pairs, $f[k] \leftrightarrow F(\Omega)$, and DTFT properties are available to assist in obtaining the transform and its inverse.
- If a causal signal $f[k]$ is absolutely summable (i.e., $\sum_{k=-\infty}^{\infty}|f[k]| < \infty$), and has a Z-transform $F(z)$, then (and only then) the DTFT of $f[k]$ can be obtained from $F(z)$ as $F(\Omega) = F(z)|_{z=e^{j\Omega}}$
- The frequency response function $H(\Omega)$ of a stable LTIDT system can be obtained from the transfer function $H(z)$ by simply replacing z by $e^{j\Omega}$.

- The DTFT relationship to the Z-transform is very similar to that of the relationship of the (CT) Fourier transform to the Laplace transform.
- The DTFT cannot handle initial conditions like the Z-transform. Therefore, it cannot be used to obtain the $y_{zi}[k]$ component of the response. However, if the system is stable, the DTFT can be used to obtain the zero-state response according to the formula: $y_{zs}[k] = F^{-1}\{H(\Omega)F(\Omega)\}$. Here, the notation $F^{-1}\{\}$ stands for the inverse DTFT.
- If $f[k]$ is a constant, an everlasting sinusoid or (more generally) an everlasting periodic signal then $F^{-1}\{H(\Omega)F(\Omega)\}$ gives the steady-state response, $y_{ss}[k]$.
- The response of a stable system to a periodic $f[k]$ that is expressed in terms of its trigonometric Fourier series, $f[k] = C_0 + \sum_{i=1}^{m} C_i \cos(\Omega_i k + \theta_i)$ is

$$y_{ss}[k] = H(0)C_0 + \sum_{i=1}^{m} |H(\Omega_i)| C_i \cos[\Omega_i k + \angle H(\Omega_i) + \theta_i]$$

- The DTFT is of fundamental importance in the design of recursive (digital) filters. It is also applicable to the analysis of truncation error in numerical algorithms.