

Experiment 5

Analog Circuit Realization of the Lorenz Chaotic System

Systems and Signals Laboratory

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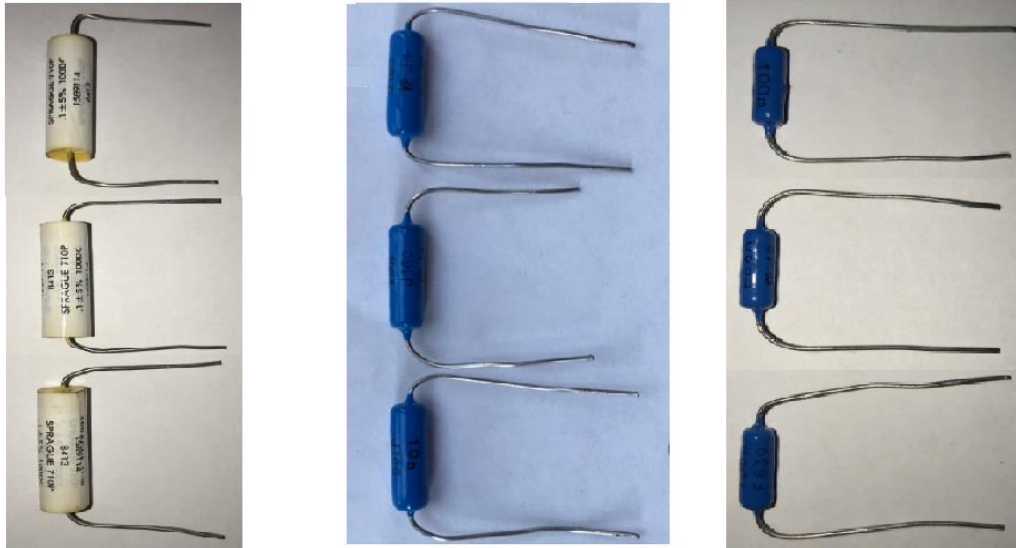
Contents:

- Pre-lab
- Lab activities: Build an analog circuit consisting of amplifiers, integrators and multipliers that implements the diffusion-less Lorenz dynamical system, and experimentally generate its phase-plane dynamics.
- Post-lab
- Links to resources and data sheets:
 - [The OPA277 op amp data sheet](#)
 - [The AD633 analog multiplier data sheet](#)
 - The Lorenz system explained: [Movie](#)
 - Circuit realization of the Lorenz system (by Paul Horowitz, Harvard University): [Movie](#)
 - Experimental setup and demonstration (by Prof. M. Hassoun): [Movie](#)

Electronic components required for this experiment

- OPA277 precision operational amplifier (5)
- AD633 analog multiplier (2)
- 10k Ω (1/4 Watt, 1%) resistor (2)
- 100 k Ω (1/4 Watt, 1%) resistor (1)
- 1M Ω (1/4 Watt, 1%) resistor (3)
- 10M Ω (1/4 Watt, 1%) resistor (3)
- 5.1k Ω (1/4 Watt, 5%) resistor (1)
- 10k Ω (1/4 Watt, 5%) resistor (1)
- 50k Ω , 10-turn precision potentiometer
- 100pF (Polypropylene film, 1%) capacitor (3)
- 0.01 μ F (Polypropylene film, 5%) capacitor (3)
- 0.1 μ F (Polypropylene film, 5%) capacitor (3)

0.1 μ F (Polypropylene film, 5%) capacitor 0.01 μ F (Polypropylene film, 5%) capacitor 100pF (Polypropylene film, 1%) capacitor



1. Pre-lab

(Source: Wikipedia) Edward Lorenz was born in West Hartford, Connecticut. He studied mathematics at both Dartmouth College and Harvard University. From 1942 until 1946, he served as a meteorologist for the United States Army Air Corps. After his return from World War II, he decided to study meteorology. Lorenz earned two degrees in the area from the Massachusetts Institute of Technology where he later was a professor for many years.

During the 1950s, Lorenz became skeptical of the appropriateness of the linear statistical models in meteorology, as most atmospheric phenomena involved in weather forecasting are non-linear. His work on the topic culminated in the publication of his 1963 paper "Deterministic Non-periodic Flow" in *Journal of the Atmospheric Sciences*, and with it, the foundation of chaos theory. He states in that paper:

“Two states differing by imperceptible amounts may eventually evolve into two considerably different states ... If, then, there is any error whatever in observing the present state—and in any real system such errors seem inevitable—an acceptable prediction of an instantaneous state in the distant future may well be impossible....In view of the inevitable inaccuracy and incompleteness of weather observations, precise very-long-range forecasting would seem to be nonexistent.”

Edward Lorenz's first weather model exhibited chaotic behavior, but it involved a set of 12 nonlinear differential equations. Lorenz decided to look for complex behavior in an even simpler set of equations, and was led to the phenomenon of rolling fluid convection. The physical model is simple: place a gas in a solid rectangular box with a heat source on the bottom.

Lorenz simplified a few fluid dynamics equations (called the Navier-Stokes equations) and ended up with a set of three nonlinear equations:

$$\frac{dx}{dt} = 10(y - x)$$

$$\frac{dy}{dt} = 28x - y - xz$$

$$\frac{dz}{dt} = xy - \frac{8}{3}z$$

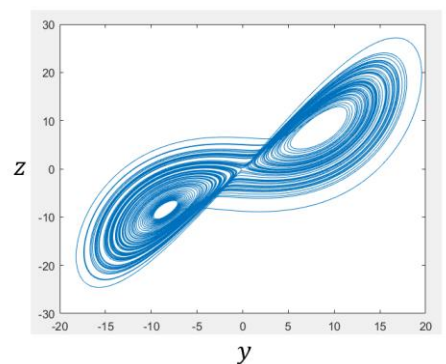
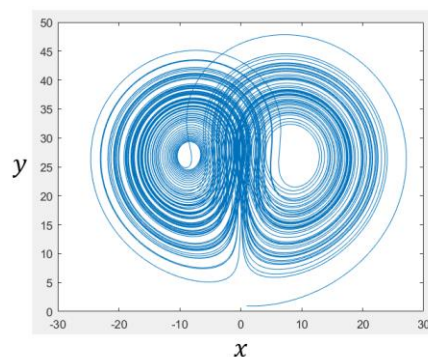
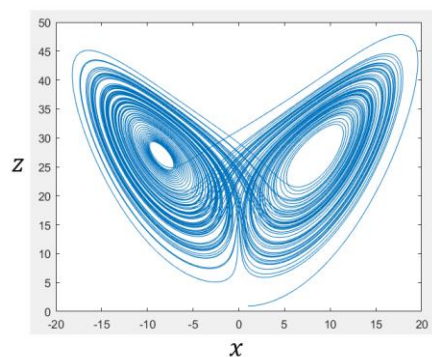
On the surface these three equations seem simple to solve. However, they represent an extremely complicated dynamical system. The following is a Matlab function that solves such system of differential equations.

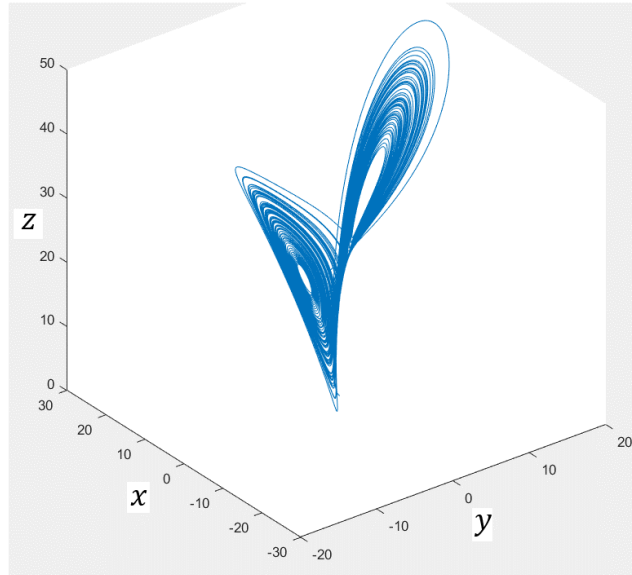
```

1 function lorenz_full
2 function fder=fun(t,u)
3     fder=[10*(u(2)-u(1)); 28*u(1)-u(2)-u(1)*u(3); u(1)*u(2)-(8/3)*u(3)];
4     end
5     options=odeset('AbsTol',1e-8,'RelTol',1e-6);
6     [t,u]=ode45(@fun, [0 100],[1;1;1],options);
7     figure(1)
8     plot(t,u(:,1),t,u(:,2),t,u(:,3))
9     figure(2)
10    plot(u(:,1),u(:,2))
11    figure(3)
12    plot(u(:,1),u(:,3))
13    figure(4)
14    plot(u(:,2),u(:,3))
15    figure(5)
16    plot3(u(:,1),u(:,2),u(:,3));
17    end

```

The following plots are the solutions to $x(t)$, $y(t)$ and $z(t)$ depicted as phase plane trajectories [for the initial state (1 1 1), and $0 \leq t \leq 100$]:





Videos to watch:

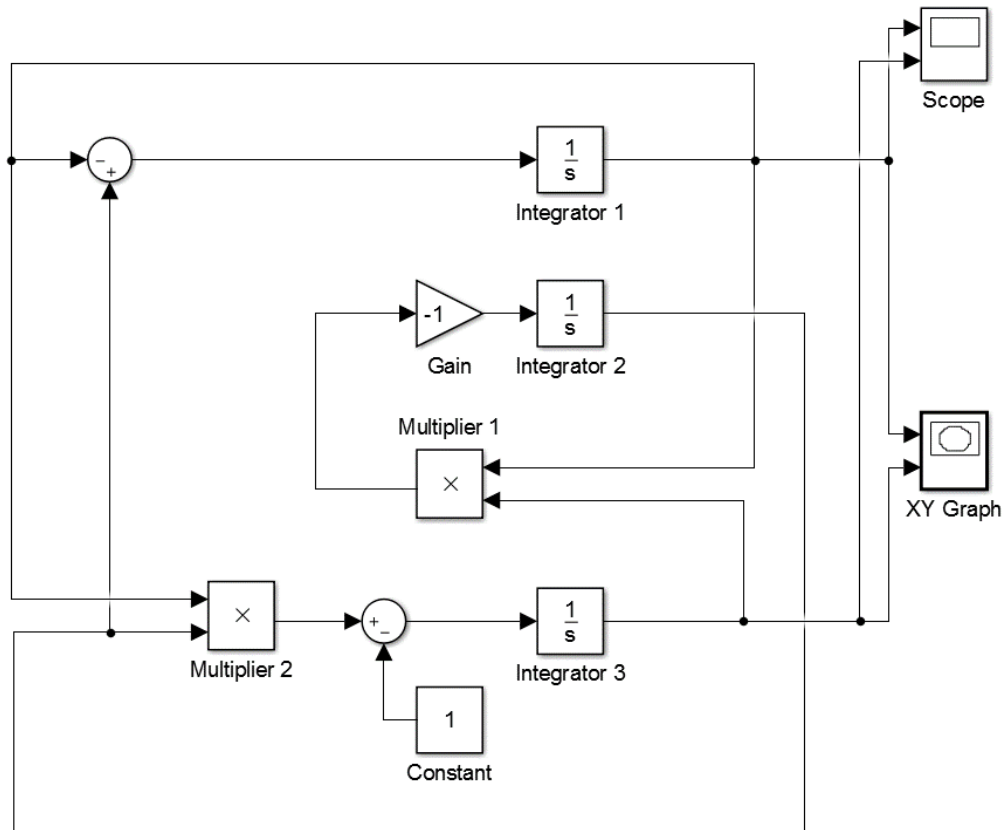
- Video explaining the dynamics of the Lorenz system: [Movie](#).
- Video demonstration of a circuit realization of the Lorenz system (by Prof. Paul Horowitz, Harvard University, Department of Physics): [Movie](#).
- Video of the realization of a Lorenz-like system on a vintage analog computer (by Prof. Mohamad Hassoun): [Movie](#).

For the lab portion of this experiment, you will build an analog circuit that solves a special form of the Lorenz system, known as the “diffusion-less Lorenz system”:

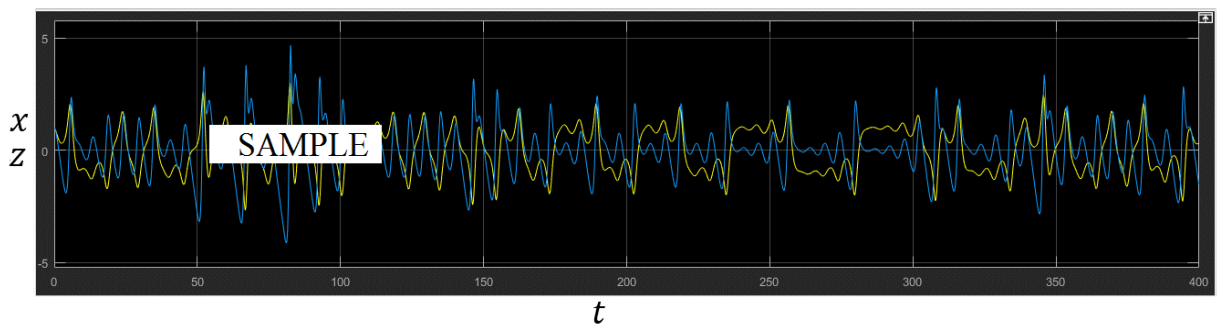
$$\begin{aligned}
 \frac{dx}{dt} &= y - x & x &= \int (y - x) dt \\
 \frac{dy}{dt} &= -xz & \text{or in integral form} \rightarrow y &= \int -xz dt \\
 \frac{dz}{dt} &= xy - a & z &= \int (xy - a) dt
 \end{aligned}$$

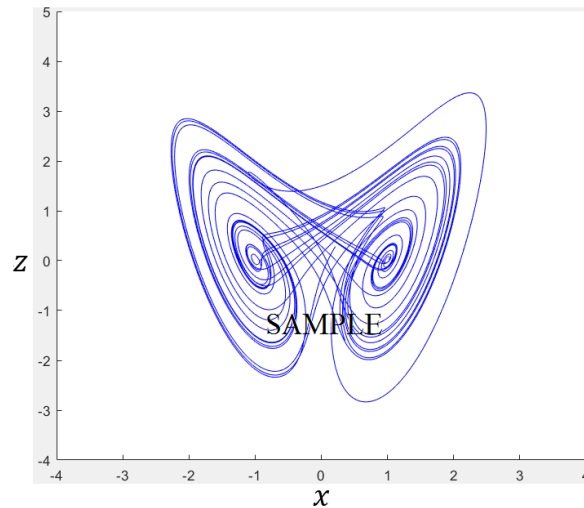
The following video shows an experimental setup and demonstration (by Prof. M. Hassoun) of such a circuit: [Movie](#). Also, the chaotic behavior of the circuit is demonstrated [here](#).

As for the pre-lab work, you are asked to complete the following simulations. A block diagram Simulink realization for the above diffusion-less Lorenz system is shown below.

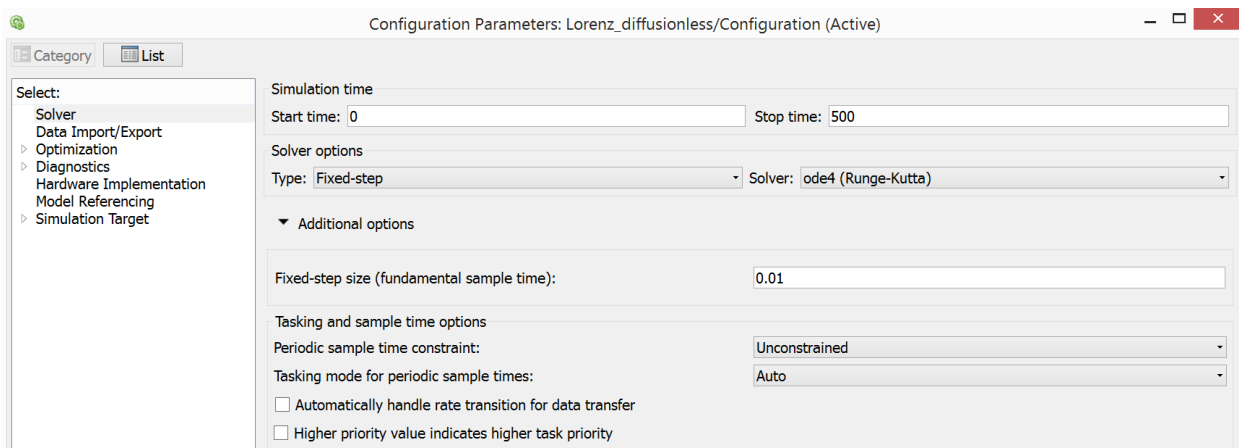


- Verify that the above block diagram realizes the diffusion-less Lorenz system.
- Use Matlab's Simulink to duplicate the above system. Set $a = 1$, integrators initial conditions to $(x(0), y(0), z(0)) = (1, 1, 1)$ and set the simulation time to 400 sec. Generate a plot of $x(t)$ and $z(t)$ versus time. Also, generate phase-plane trajectories for $z(t)$ vs $x(t)$, $y(t)$ vs $x(t)$ and $z(t)$ vs $y(t)$. See sample results shown in the figures below.

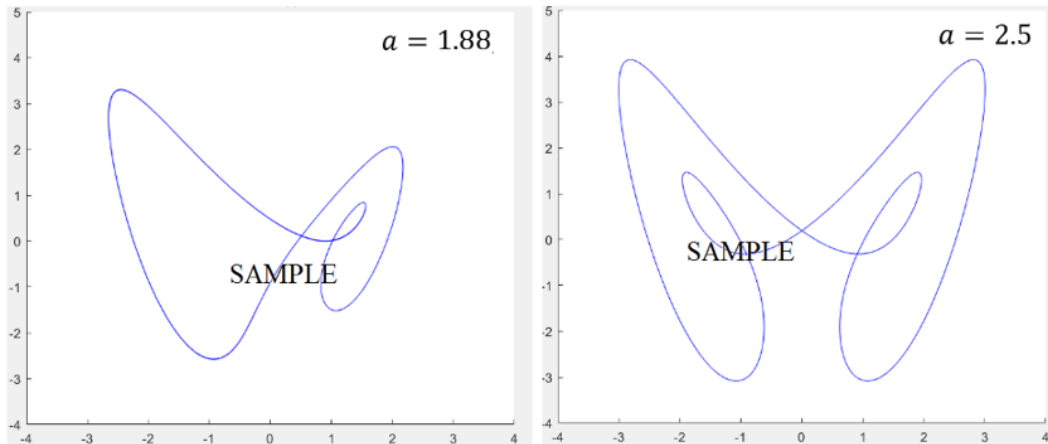




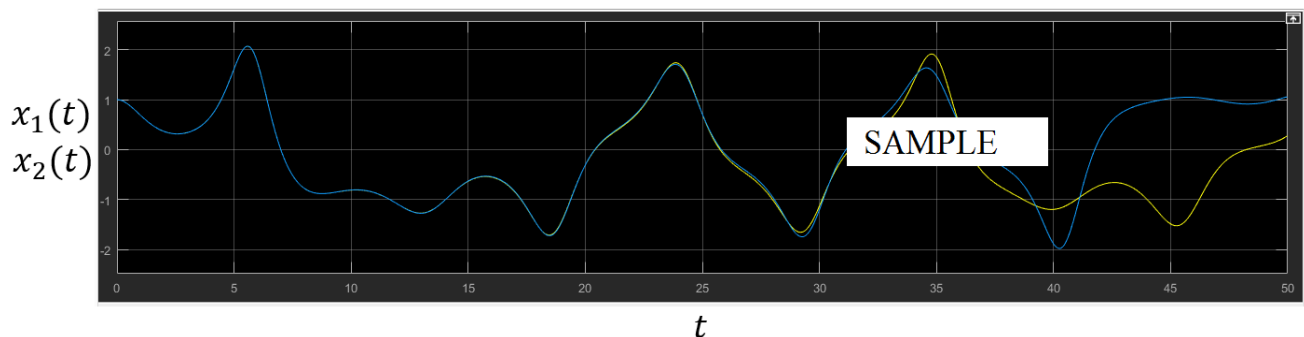
Note: Make sure to configure Simulink's Solver (under the Menu: Simulation/Model Configuration Parameters) as depicted in the following figure. This allows for smooth trajectories and slows down the simulation so that you can visualize the evolution of the solution.



- c. Repeat the simulations with $a = 1.88$ and $a = 2.5$ after setting integrator initial conditions to $(1,0,0)$ and setting the simulation time to 500. Generate $z(t)$ vs $x(t)$ plots for each case. Your simulations will show how the dynamics exhibit an initial transient, but ultimately converge to a sustainable periodic trajectory (limit-cycle). Simulink will automatically erase the initial transient trajectory and keep the periodic trajectory. The following figures depict sample simulation results.



- d. Employ Simulink to demonstrate the chaotic behavior of the system for $a = 1$. You can do that by simulating two systems that are identical except that one starts from the initial state $(1,1,1)$ and the other starts from $(1.001,1,1)$. You should generate a plot of $x_1(t)$ and $x_2(t)$ similar to the following plot (set simulation time to 50). Hint: you may get the two scope traces separately and then use Microsoft Paint (or an equivalent image editor) to cut and paste to generate the plot shown below.



It is important to read the following section (as part of the pre-lab assignment) in order to familiarize yourself with the upcoming lab activity.

2. Lab activity

Make actual measurements of all resistor and capacitor values that you use in the following circuit. Make sure to take photographs of all circuits that you build in the lab. Also, you are to save all scope trace images. You will be using these photos/images in your lab report.

This experiment is centered on a nonlinear electronic circuit that realizes the diffusion-less Lorenz system. The precision OPA277 operational amplifier will be used to build integrators, amplifier and inverter circuits. You will also use two AD633 analog multipliers to realize the required xz and xy products.

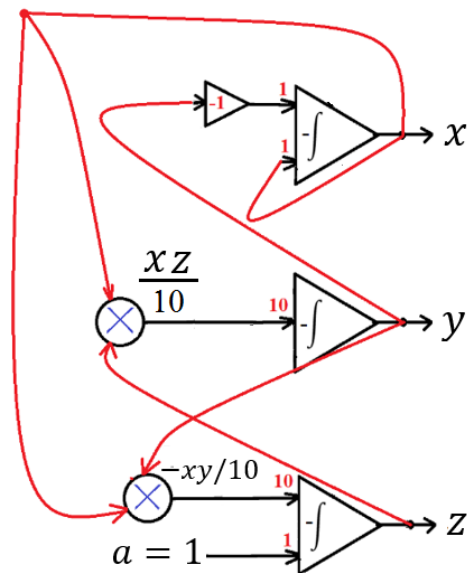
The circuit solves the following set of simultaneous integral equations:

$$x = \int (y - x) dt$$

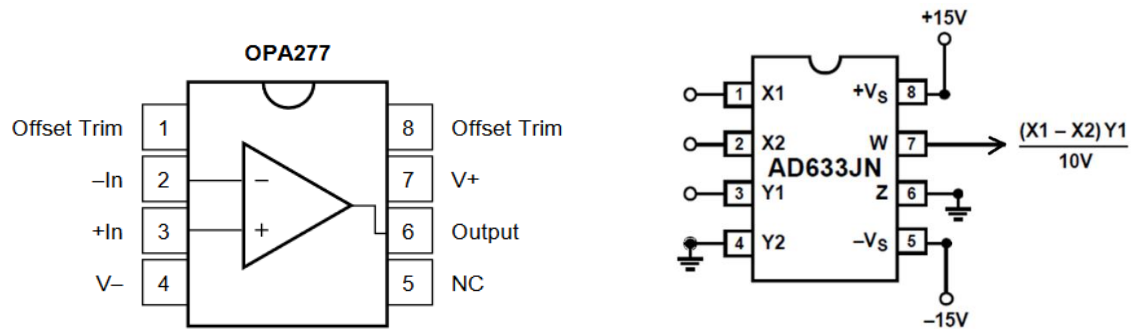
$$y = \int -xz dt$$

$$z = \int (xy - a) dt$$

The following block diagram represents the basis for the circuit realization:

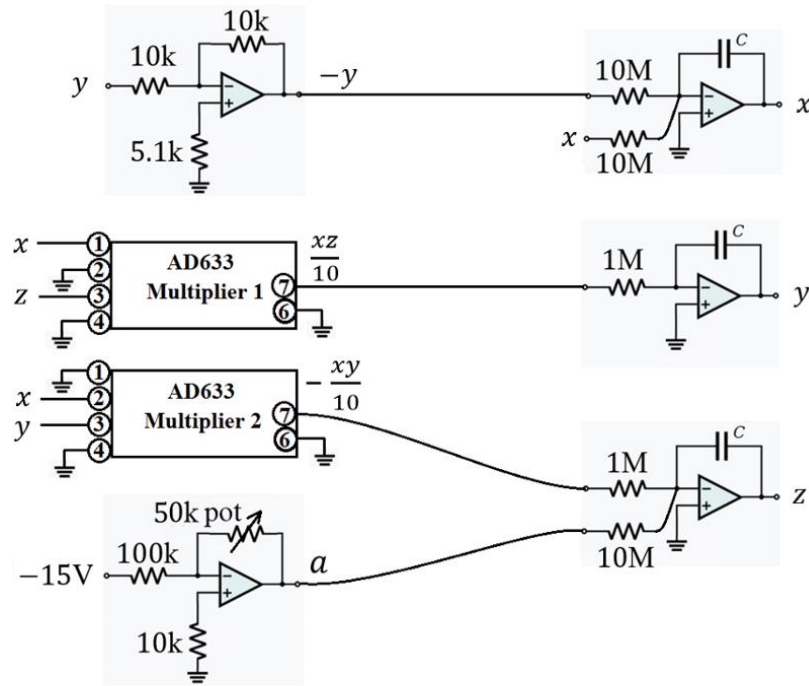


For convenience, the pin diagrams for the OPA277 and AD633 are shown below.



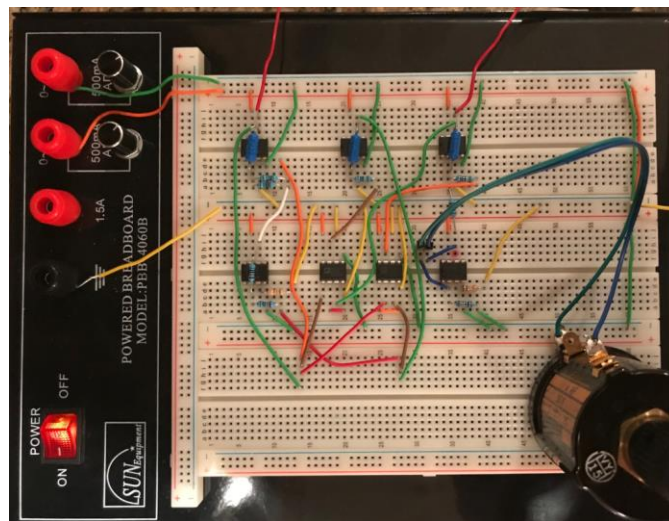
Note how the AD633 (with pins 4 and 6 grounded) can generate the scaled product $\frac{xy}{10}$ (pin 7) if pin 2 is grounded and x and y are connected to pins 1 and 3, respectively. Grounding pin 1 and connecting x to pin 2 and y to pin 3 generates $-\frac{xy}{10}$ on pin 7.

- a. Use the breadboard to build the following circuit. Use three 100pF, 1% tolerance Polypropylene capacitors. Start by inserting the chips strategically (and carefully) on the breadboard.



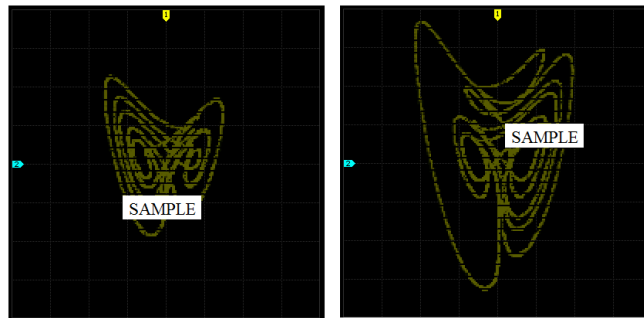
Make sure that all seven chips have the proper $\pm 15V$ supply connections

A sample bread-boarded circuit:

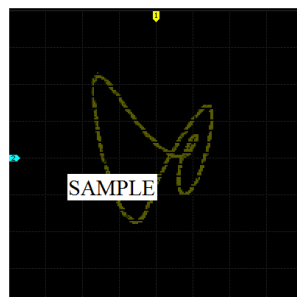


You are to ask your lab instructor to double check your circuit wiring before applying power.

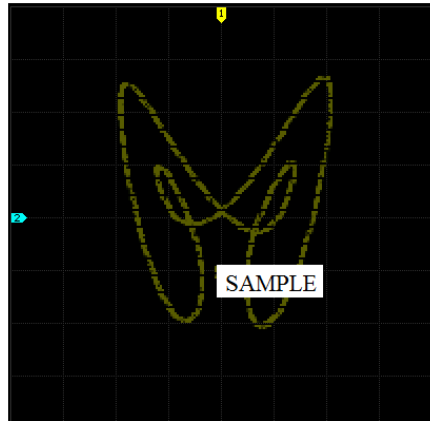
- b. Connect the output signal $x(t)$ to CH1 of the scope. Connect the output signal $z(t)$ to CH2 of the scope. Set both channels to 1.5V/div. Set the time scale to 5ms/div. Set the scope to the X-Y (full display) plotting mode. Power up the circuit. The noise in the circuit will be exploited in order to trigger the dynamics of the system being simulated.
- c. Use a voltmeter to monitor the voltage that sets the parameter a . This is the output of the inverting amplifier with the 50k Ω precision potentiometer. Adjust the pot so that the voltmeter reads 1 Volt. The scope should display patterns similar to the ones shown below. Save three captures of the scope's traces on your memory stick.



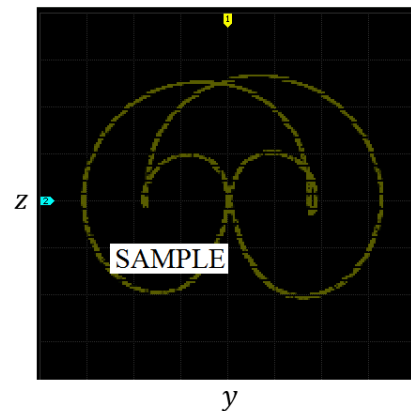
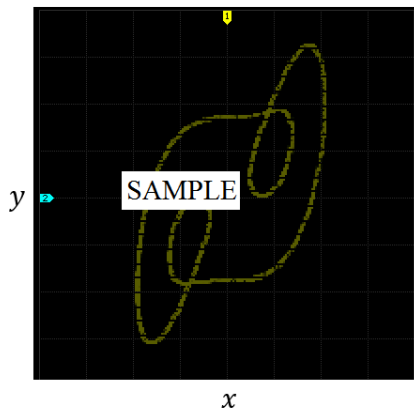
- d. Switch the circuit power off. Replace the integrator capacitors by 0.01 μ F (5%) Polypropylene film capacitors. Power up your circuit. What effect does using the larger capacitors have on the solution? You need to explain these effects in your lab report. Save the scope's traces on your memory stick.
- e. Repeat the previous step using 0.1 μ F (5%) Polypropylene film capacitors. Watch the [video](#) of the circuit dynamics for this case captured in real-time by a vintage X-Y plotter.
- f. Go back and reinstall the 100pF integrator capacitors. Adjust the 50k Ω potentiometer so that you set a close to 1.85. Fine tune the 50k Ω precision potentiometer until a stable periodic orbit (similar to the one show below, or similar to its mirror image w.r.t. the vertical axis) is generated. Record the value of a . Save the scope's trace image.



- g. Switch the power off, then switch it back on. Does the circuit converge to the same limit-cycle? Does the trajectory agree with the Pre-lab Simulink simulations? Explain.
- h. Repeat Steps f and g for $a = 2.5$. Fine tune the $50\text{k}\Omega$ pot until a stable periodic orbit (similar to the one show below) is generated. Save the scope's trace image.

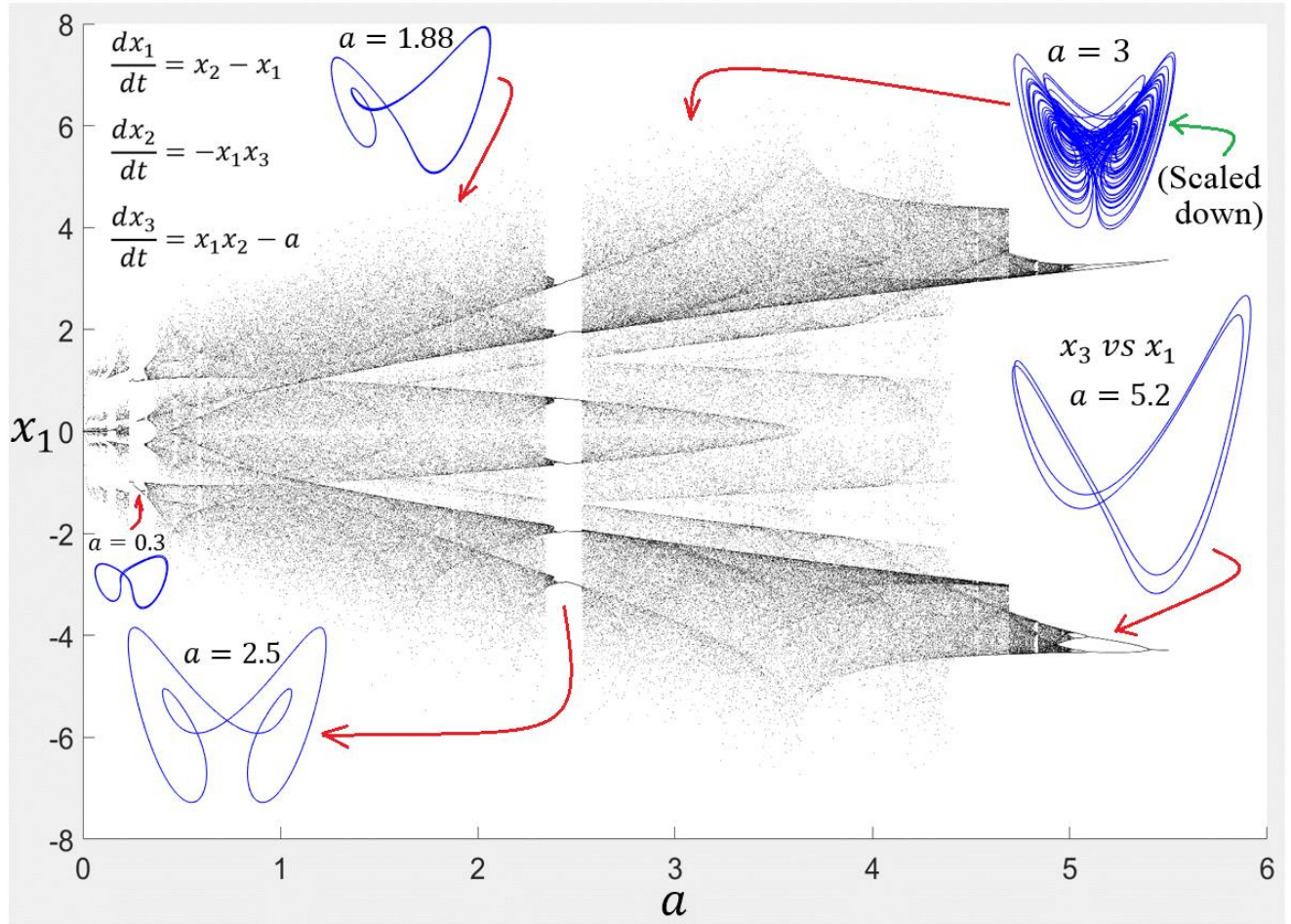


- i. With a tuned as in Step h, generate and save the phase plane traces for $y(t)$ vs $x(t)$. Repeat for $z(t)$ vs $y(t)$. The following figure shows samples.



The diagram shown below is a plot of all local maxima and local minima of $x(t)$ when t is very large. This plot was computed by solving the Lorenz system of differential equations numerically for many values of the parameter a from zero to six. The plot is called a *bifurcation diagram*. The shaded areas depict the values of a for which the solutions $x(t)$, $y(t)$ and $z(t)$ don't oscillate. The plot also shows examples of oscillatory solutions $[z(t) \text{ vs } x(t)]$ for $a = 0.3, 1.88, 2.5$ and 5.2 . Obtain and save the plots for $z(t) \text{ vs } x(t)$ and $y(t) \text{ vs } x(t)$ with $a \cong 5.2$. Finally, try to obtain the plot $z(t) \text{ vs } x(t)$ for the oscillations (that are difficult to get) in the vicinity of $a \cong 0.4$ and $a \cong 0.55$. Replace the $100\text{k}\Omega$ resistor with a $1\text{M}\Omega$ resistor for increased resolution and control over the parameter a .

Bifurcation Diagram of the (Diffusion-less) Lorenz Dynamical System



As you disassemble your circuit, and in case you forgot to measure element values, carefully measure and record the actual resistor, pot and capacitors value.

3. Post-lab

Write your lab report according to the required format. Analyze and discuss, in details, all results obtained from your pre-lab and your lab activity. Compare the theoretical and experimental dynamics of the system. Report the absolute relative errors (in percent) for all experimental measurement whenever the true value is known.