

Experiment 7

Fourier Series of Periodic Signals and Filtering

Systems and Signals Laboratory

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Electronic instruments and components required for this experiment

- Krohn-Hite 3202 dual variable filter
- LF351 wideband operational amplifier
- 10k Ω (1/4 Watt, 1%) resistor (3)

Background

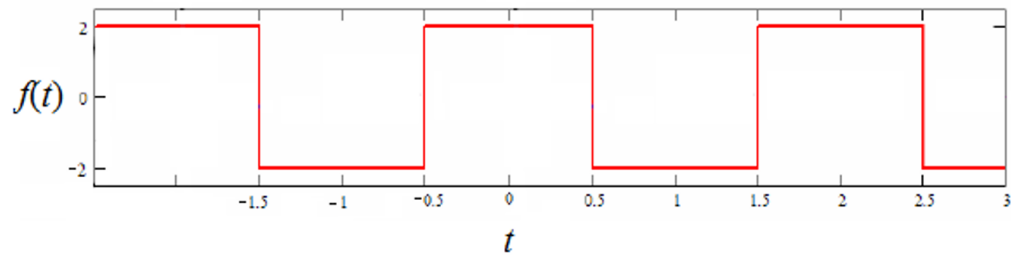
The Fourier series represents a periodic signal $f(t)$ (with period T_o and fundamental frequency $\omega_o = \frac{2\pi}{T_o} = 2\pi f_o$) as an infinite sum of sinusoidal signals having harmonic (integer multiples of ω_o) frequencies. The series has important applications in linear system steady-state analysis (thanks to the superposition principle). The compact trigonometric Fourier series can be expressed as,

$$f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_o t + \theta_n)$$

In practice, the signal $f(t)$ is a physical (real-valued, laboratory generated) signal. For physical periodic signals we have,

$$\lim_{n \rightarrow \infty} C_n = 0$$

Example 3. The Fourier series of the following square signal (with $T_o = 2$ and $\omega_o = \frac{2\pi}{T_o} = \pi$)

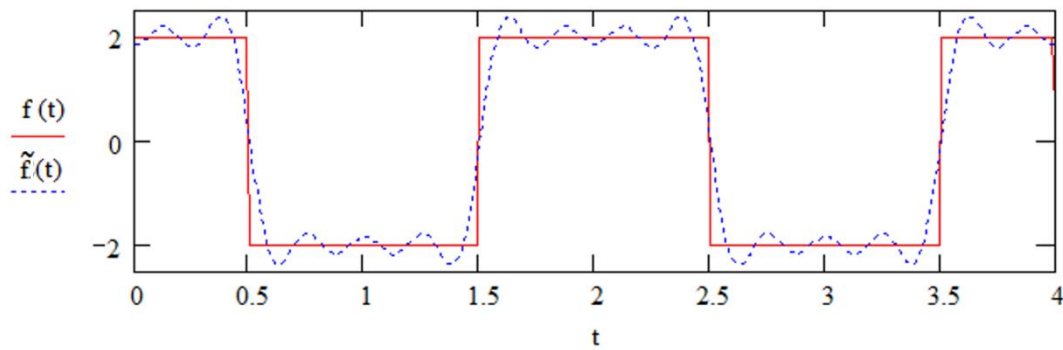


is given by the series (obtained from a [table](#)),

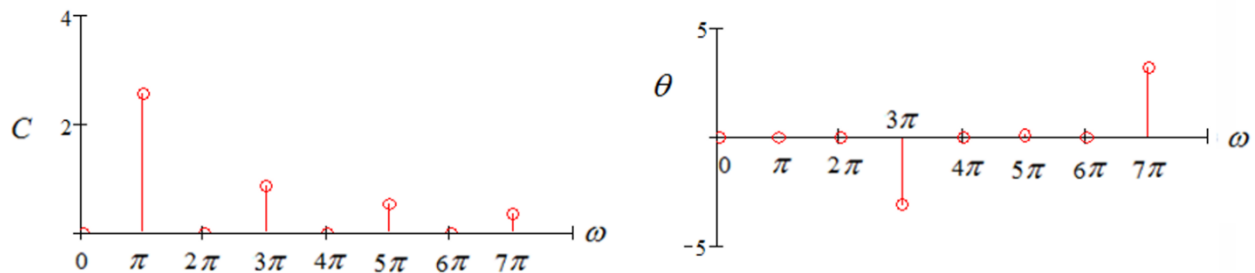
$$f(t) = \frac{8}{\pi} \left(\cos(\pi t) + \frac{1}{3} \cos(3\pi t \pm \pi) + \frac{1}{5} \cos(5\pi t) + \frac{1}{7} \cos(7\pi t \pm \pi) + \frac{1}{9} \cos(9\pi t) + \dots \right)$$

It should be noted that the Fourier series coefficients C_0 , C_n and θ_n are not affected by the value of the signal's frequency. For this particular signal, the signal average $C_0 = 0$. Also, because of symmetries in the signal, all even harmonics are zero.

The following is a plot of the signal $f(t)$ and its approximation $\tilde{f}(t)$ using the first four non-zero harmonic components (with frequencies π , 3π , 5π and 7π).



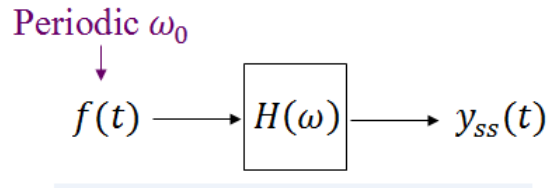
The following figure depicts the magnitude and phase spectra of $f(t)$.



The Fourier series representation of a signal allows for a very convenient method for determining the steady-state response of linear filters, as illustrated in the following example.

General Formula for ac Steady-State Analysis with Periodic Inputs

Given a filter with a frequency response function $H(\omega)$ and input $f(t)$, find $y_{ss}(t)$



For compact Fourier series representation of $f(t)$,

$$f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$$

The formula for the (approximate) steady-state response of the filter is given by

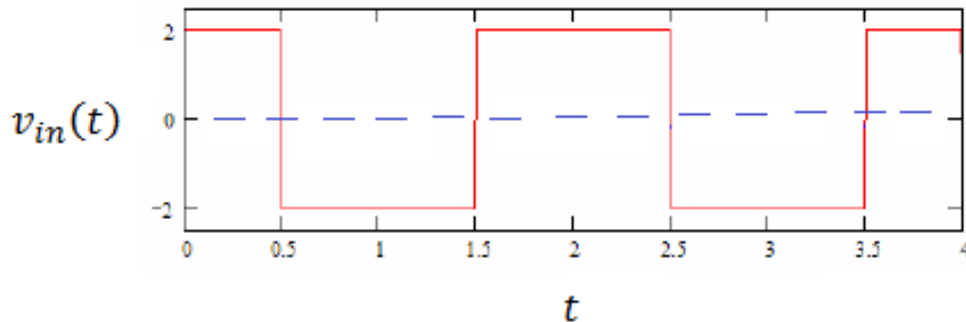
$$y_{ss}(t) \cong H(0)C_0 + \sum_{n=1}^m |H(n\omega_0)| C_n \cos[n\omega_0 t + \theta_n + \angle H(n\omega_0)]$$

Good judgment should be used in determining the upper limit (m) of the sum. For example if $f(t)$ has no jump discontinuities (say a triangular wave) and the filter is low-pass, then $m = 3$ will do. On the other hand, if $f(t)$ has jump discontinuities (think square wave) and the filter is low-pass, then m would be about 10.

Example. Find the steady-state response $v_{o_{ss}}(t)$ of the filter:

$$H(s) = \frac{V_o(s)}{V_{in}(s)} = \frac{2}{s^2 + 3s + 4}$$

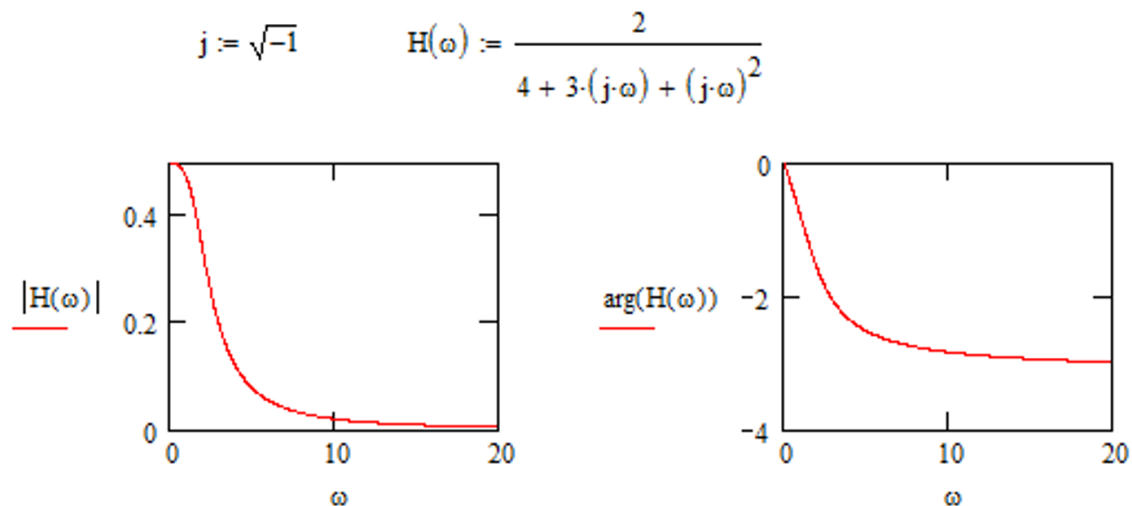
Assume that the input voltage, $v_{in}(t)$, is the square wave shown in the figure below.



The poles of $H(s)$ are $-\frac{3}{2} \pm j\frac{\sqrt{7}}{2}$. They have negative real part. Therefore, the filter is stable and the frequency response function for ac steady-state operation is given by

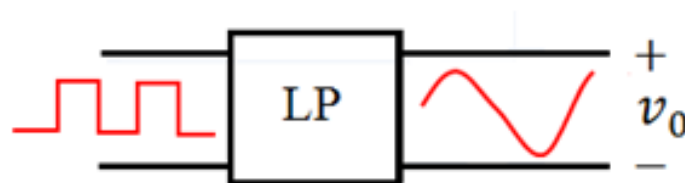
$$H(\omega) = H(s)|_{s=j\omega} = \frac{2}{-\omega^2 + j3\omega + 4}$$

A Mathcad generated plot of the magnitude and phase spectra of the filter are shown in the figure below. It shows a low-pass filter with a cutoff frequency, $\omega_c \approx 2\text{rad/sec}$. Notice that the input signal components that have frequencies greater than $5\omega_c = 10\text{ rad/sec}$ are highly attenuated.

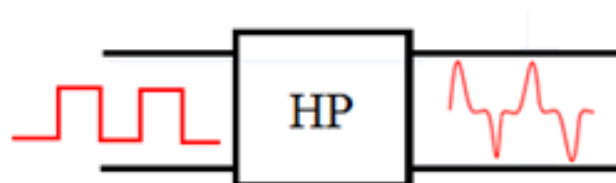


Before we perform the quantitative analysis it helps if we can predict the qualitative behaviour of the response.

Because of its low-pass characteristics, the filter will pass the frequencies below its cutoff frequency and attenuate the higher frequencies of the input signal. Knowing that the high frequency content of a square wave is due to the sharp transition in the signal (and that the low frequencies are due to slow varying portions), we can then predict that the steady-state response will look as follows (the filter rounds the sharp corners):



Had the circuit been a high-pass filter circuit, then the steady-state response to a square wave would have looked as follows (sharp transitions are retained and slow varying portions of the signal are blocked):



The filter output, $v_{o_{ss}}(t)$, is given by

$$v_{o_{ss}}(t) \cong H(0)C_0 + \sum_{n=1}^m |H(n\omega_0)|C_n \cos[n\omega_0 t + \theta_n + \angle H(n\omega_0)]$$

with

$$v_{in}(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n) = \frac{8}{\pi} \cos(\pi t) + \frac{8}{3\pi} \cos(3\pi t + \pi) + \frac{8}{5\pi} \cos(5\pi t) + \dots$$

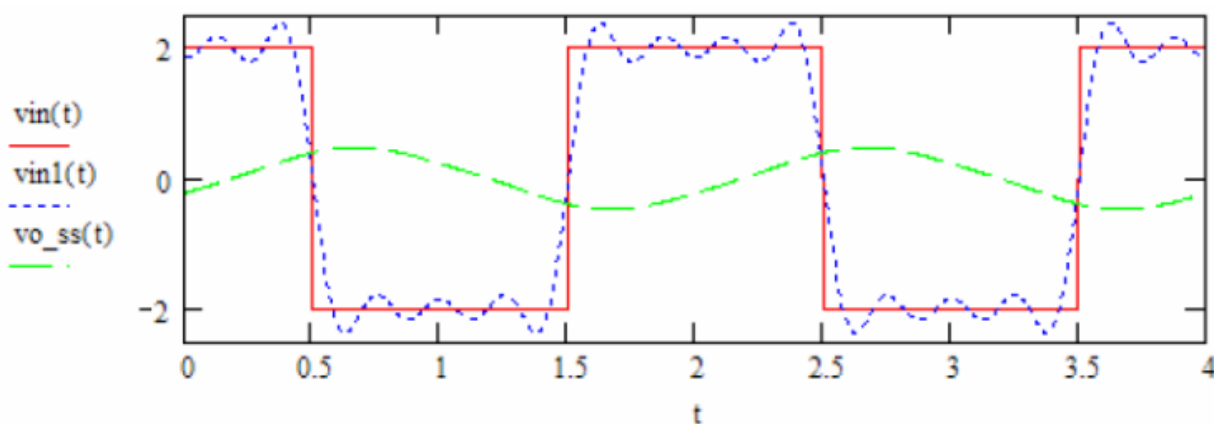
The frequency response for the first three non-zero harmonics of the input signal are given by

$$\begin{aligned} H(0) &= 0.5 \\ H(\pi) &= 0.180\angle -2.128 \\ H(3\pi) &= 0.022\angle -2.820 \\ H(5\pi) &= 0.008\angle -2.950 \end{aligned}$$

Note that there is a very significant attenuation for the 5π component. So, we may ignore it as well as all of the higher frequency components. We may then retain the first two terms in the remaining analysis without any significant loss of accuracy. Therefore, the output of the filter can be approximated as,

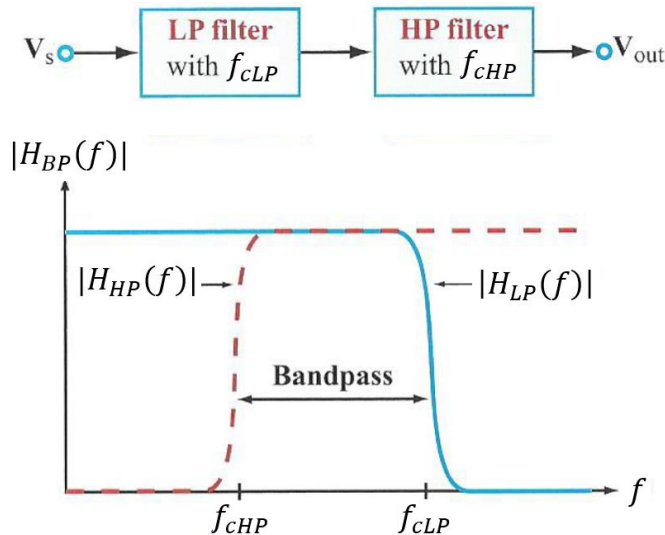
$$\begin{aligned} v_{o_{ss}}(t) &\cong H(0)C_0 + \sum_{n=1}^m |H(n\omega_0)|C_n \cos[n\omega_0 t + \theta_n + \angle H(n\omega_0)] \\ &= (0.5)(0) + (0.18) \left(\frac{8}{\pi}\right) \cos(\pi t - 2.128) + (0.022) \left(\frac{8}{3\pi}\right) \cos(3\pi t + \pi - 2.820) \\ &= 0.458 \cos(\pi t - 2.128) + 0.019 \cos(3\pi t + 0.322) \end{aligned}$$

The following figure depicts the input square wave, the approximated input (as a truncated Fourier series) and the approximated steady-state output.



Band-Pass Filter Realization as the Cascade of a Low-Pass and a High-Pass Filters

A band-pass filter $H_{BP}(s)$ can be achieved by cascading a low-pass filter and a high-pass filter, $H_{BP}(s) = H_{LP}(s)H_{HP}(s)$, as illustrated in the figure below. It is important to note that the band-pass lower cutoff frequency is the high-pass filter's cutoff frequency f_{CHP} . Also, the band-pass upper cutoff frequency is that of the low-pass filter's cutoff frequency f_{CLP} (with $f_{CLP} > f_{CHP}$).



1. Pre-Lab

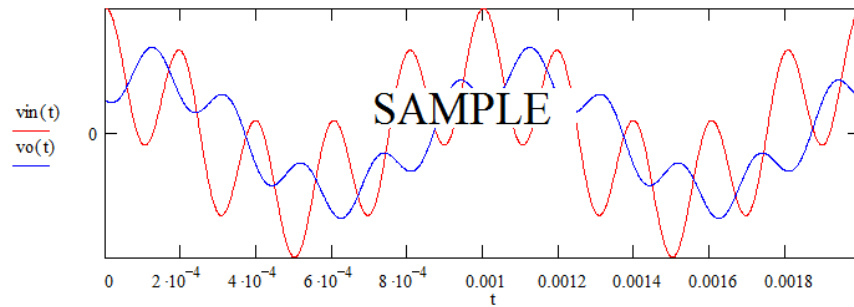
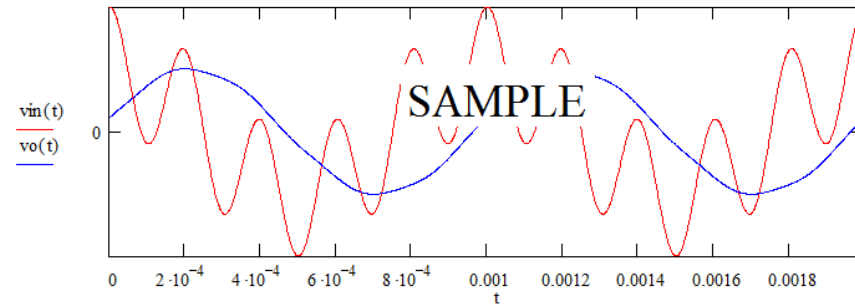
Study the above “Background” section which should help you complete the following steps.

- Read the [brochure](#) and pages 1-9 of the Krohn-Hite 3202 dual variable filter [user's manual](#).
- Determine and plot the ac steady-state response of the 4th order low-pass Butterworth filter with cutoff frequency $f_c = 1.5\text{kHz}$ (remember that $\omega = 2\pi f$)

$$H(f) = \frac{V_o(f)}{V_{in}(f)} = \frac{1}{\left(\frac{f}{f_c}\right)^4 - j2.61\left(\frac{f}{f_c}\right)^3 - 3.41\left(\frac{f}{f_c}\right)^2 + j2.61\left(\frac{f}{f_c}\right) + 1}$$

to the input signal $v_{in}(t) = 2 \cos(2,000\pi t) + 2 \cos(10,000\pi t)$. This signal is the sum of two sinusoids: The first sinusoid is 4Vpp with frequency of 1kHz and the second sinusoid is 4Vpp with frequency of 5kHz. You may use Mathcad to perform the calculations.

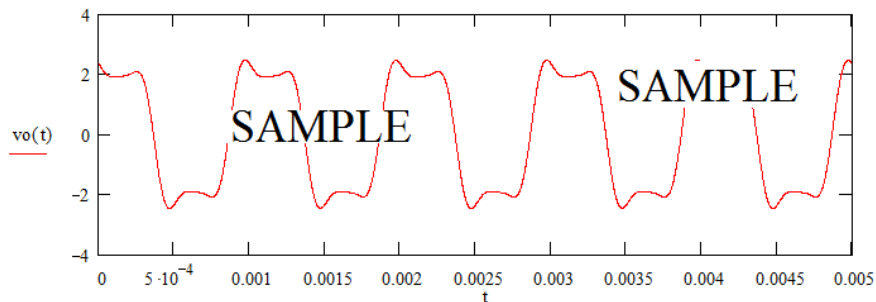
- Repeat Step b for $f_c = 2\text{kHz}$, 4kHz and 8kHz. The solution for the cases of $f_c = 2\text{kHz}$ and 4kHz are shown below.



- d. Repeat Step b for a 50% duty cycle, 4Vpp and $f_o = 1\text{kHz}$ square wave input signal and a filter cutoff frequency: $f_c = 1.5\text{kHz}$, 4kHz and 8kHz. Employ the formula,

$$v_{o_{ss}}(t) \cong H(0)C_0 + \sum_{n=1}^{10} |H(2\pi n f_0)| C_n \cos[2\pi n f_0 t + \theta_n + \angle H(2\pi n f_0)]$$

The solution for the case of $f_c = 4\text{kHz}$ is shown below.

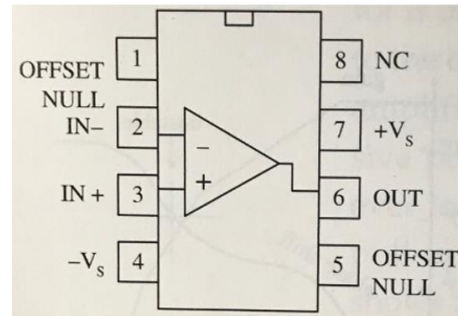
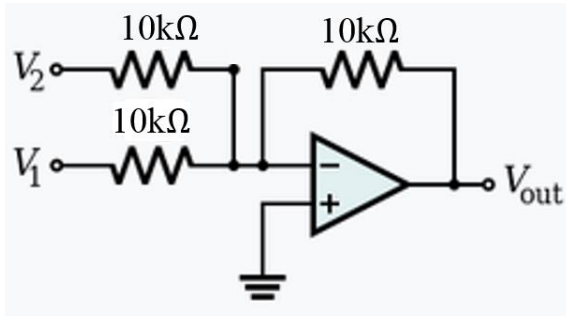


- e. Search your laptop for the folder that contains the configuration file (for WaveForms) “default.dwf3work” and then download “[experiment7](#)” and “[experiment7-filter](#)” configuration files and place them inside that folder. You will need these files for the Lab activity.

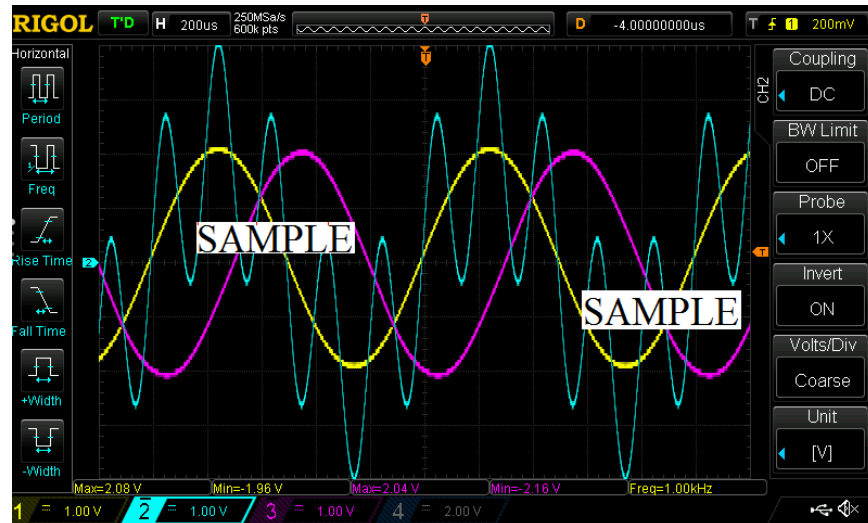
It is important to read the following section (as part of the pre-lab assignment) in order to familiarize yourself with the upcoming lab activity.

2. Lab-Activity

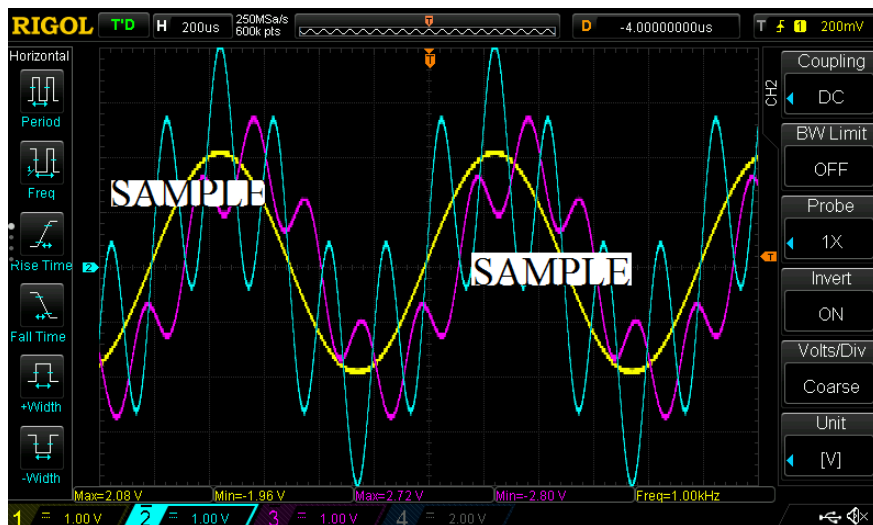
- a. Build the inverting adder circuit shown below. Set V_1 and V_2 to 4Vpp sinusoidal signals. Set the frequency of V_1 to 1kHz and that of V_2 to 5kHz. Use the RIGOL waveform generator to align the phases of the two signals. Connect V_1 to CH1 of the scope. Connect the output of the summer to CH2 of the oscilloscope. Invert CH2 in order to undo the inversion of the adder. Set the scope CH1, CH2 and CH3 to 1V/div and 200 μ sec/div.



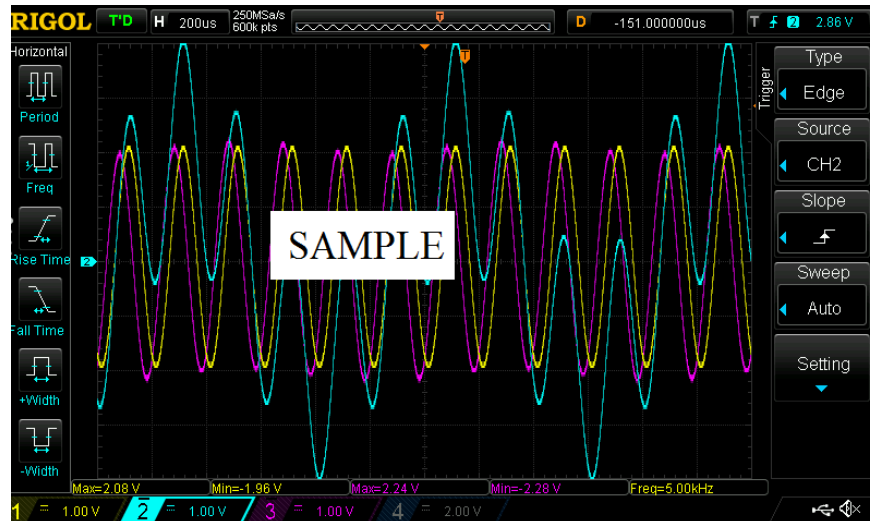
- b. Connect the output of the adder to the input of the left hand side of the Krohn-Hite 3202 filter. Connect the output of the filter to CH3 of the scope. Invert CH3 so that you correct for the adder's inversion. Set the filter to low-pass with cutoff frequency of 1.5kHz; the filter is now a 4th order Butterworth filter with $f_c = 1.5$ kHz and is supposed to pass the 1kHz sinusoid component of the input and block the 5kHz component. Shift the three signal displayed on the scope so that the traces look like the ones shown in the following sample image. Configure the information bar of the scope to display the max and min amplitudes of the signals on CH1 and CH2 and the frequency of the signal on CH1. Save the scope's display image.



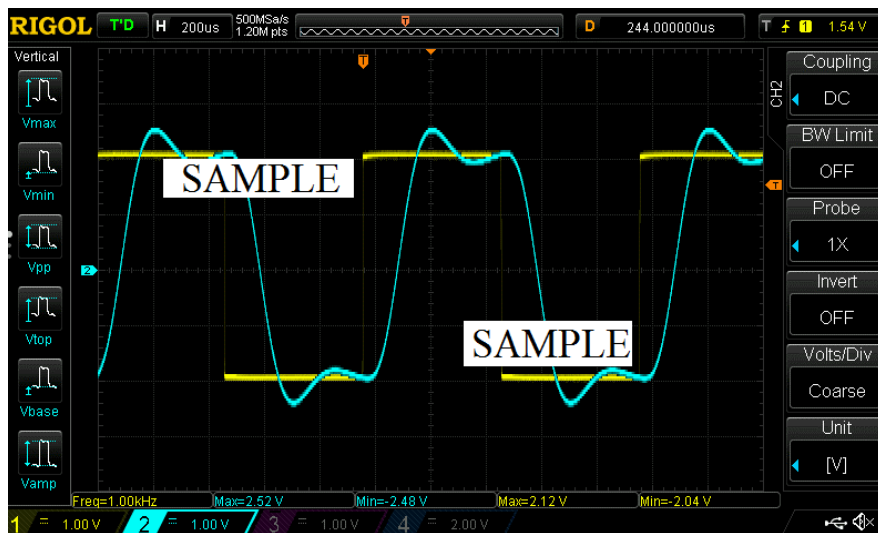
- c. Change the cutoff frequency of the filter to 2kHz and save the scope's display image. Repeat for $f_c = 4\text{kHz}$ and $f_c = 8\text{kHz}$. Save the scope's display images. The following figure shows the result for $f_c = 4\text{kHz}$.



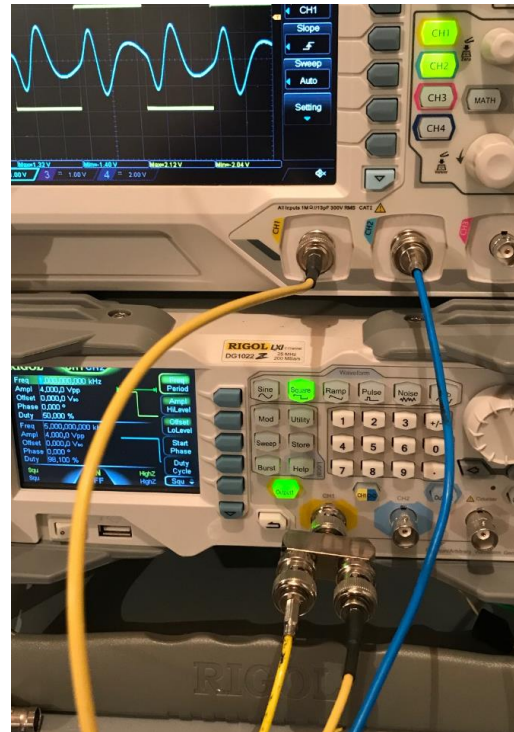
- d. Repeat Step b with the filter set to high-pass with $f_c = 2\text{kHz}$. Change the connection on CH1 of the scope from the 1kHz sinusoid to the 5kHz sinusoid. (If the scope traces are not stable, consider setting the triggering source to CH2). Save the scope's display image. The scope display should look like the one shown in the figure below. In this case, the 5kHz sinusoid is passed through, but the 1kHz sinusoid is blocked.



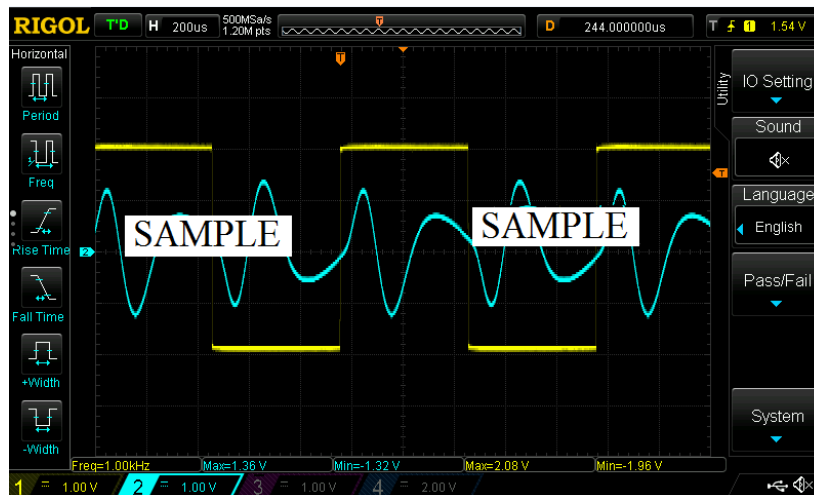
- e. Change the input of the filter to a 4Vpp, 50% duty cycle square wave with 1kHz frequency (you don't need the adder any more) and display it on CH1 of the scope. Connect the output of the filter to CH2 of the scope. Inversion should now be set to "OFF" for both CH1 and CH2. Disable CH3 of the scope. Set the filter to low-pass with $f_c = 1.5\text{kHz}$. Save the scope's display image. Repeat for $f_c = 4\text{kHz}$, 8kHz and 100kHz . The result for $f_c = 4\text{kHz}$ is shown below.



- f. Configure the Krohn-Hite 3202 as a band-pass filter with lower cutoff frequency of 2kHz and a higher cutoff frequency of 6kHz. This can be accomplished by cascading a high-pass filter with $f_c = 2\text{kHz}$ (realized by the left side of the Krohn-Hite 3202) and a low-pass filter with $f_c = 6\text{kHz}$ (realized by the right side of the Krohn-Hite 3202). Refer to the picture below.



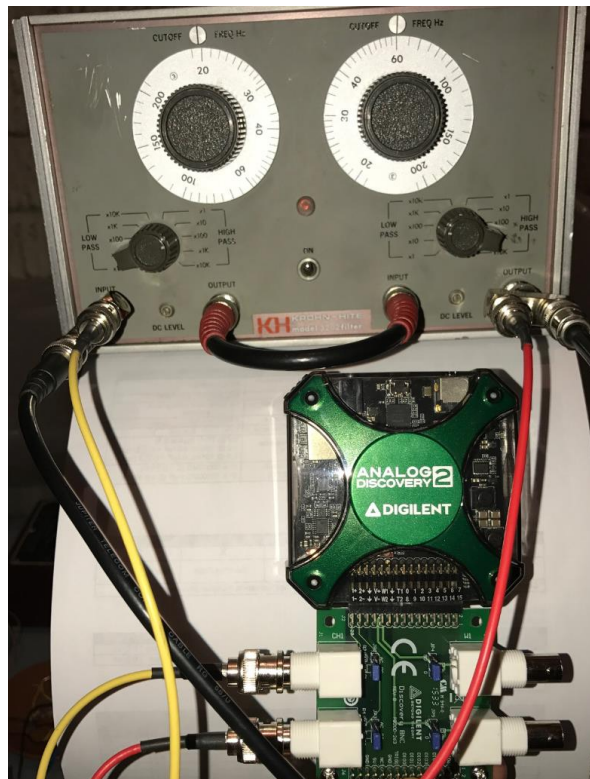
Apply a 4Vpp, 50% duty cycle and 1kHz square wave as input to the bandpass filter. Display the square wave on CH1 of the scope and the output of the filter on CH2 (use a BNC-BNC splitter adaptor as shown in the above figure). Save the scope's display image. A sample result is shown below.



- g. In this step of the experiment you will determine the spectral components (harmonic amplitudes or Fourier series C_n coefficients) of the square wave signal. And the spectral components of the signal at the output of the band-pass filter (used in Step f). You will use the Digilent AD2 module and the WaveForms user interface on your PC.

Connect the AD2 module to your PC using the supplied USB cable. Run WaveForms and select “experiment7” configuration file when prompted (you were asked to download this configuration file to the proper folder on your PC in the pre-lab activity).

Starting from the instrument setup as in Step f, use two BNC-BNC splitters to connect the input of the filter to CH1 of the AD2 module. Similarly, connect the output of the filter to CH2 of the AD2 module. Refer to the figure below for connections guide.



Click on the “Single” run button in the WaveForms screen on your PC. This should display the spectral components of the square signal (yellow spikes) and the spectral components of the filtered signal (green spikes), as shown below. Make a screen capture of the spectrum.

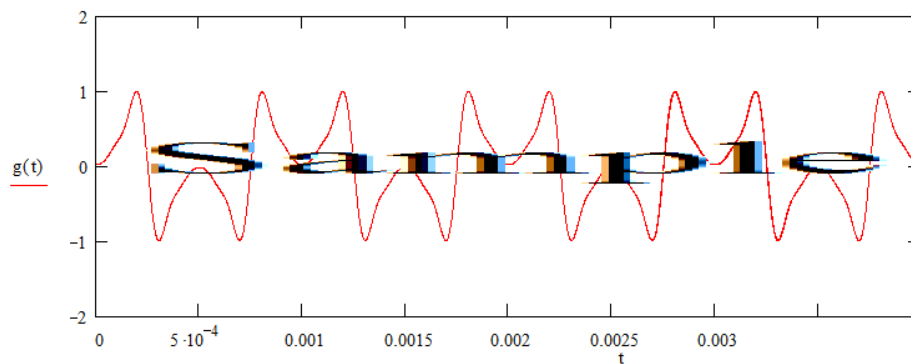


Click on the small box to the left of “Trace 2” (located on the right hand side of the WaveForms interface) to disable the green spikes. Move the mouse and position the cursor at the peak of each of the yellow spikes and record its frequency and amplitude. This will lead to five pairs of measurements. Refer to the amplitudes as $C_1, C_2, \dots, C_5$. In your lab report, you are to compare these measurements to the exact Fourier series coefficients for this signal; from the “Background Section” the exact amplitudes are given by: $\frac{8}{\pi}, \frac{8}{3\pi}, \frac{8}{5\pi}, \frac{8}{7\pi}, \frac{8}{9\pi}$.

Click on the small box next to “Trace 1” in order to disable the yellow spectra. Click of the box to the left of “Trace 2” to display the (green) spectra of the filtered signal. Move the mouse and position the cursor at the peak of each of the spikes and record the frequency and amplitude. Refer to the five amplitude measurements as $CF_1, CF_2, \dots, CF_5$. In your lab report, you are to use Mathcad to plot the signal

$$g(t) \cong (CF_1)\cos((2\pi 1000)t) + (CF_2)\cos(3(2\pi 1000)t + \pi) + (CF_3)\cos(5(2\pi 1000)t) + (CF_4)\cos(7(2\pi 1000)t + \pi) + (CF_5)\cos(9(2\pi 1000)t)$$

and compare it to the scope’s trace of the filtered signal, from Step f. Sample result:

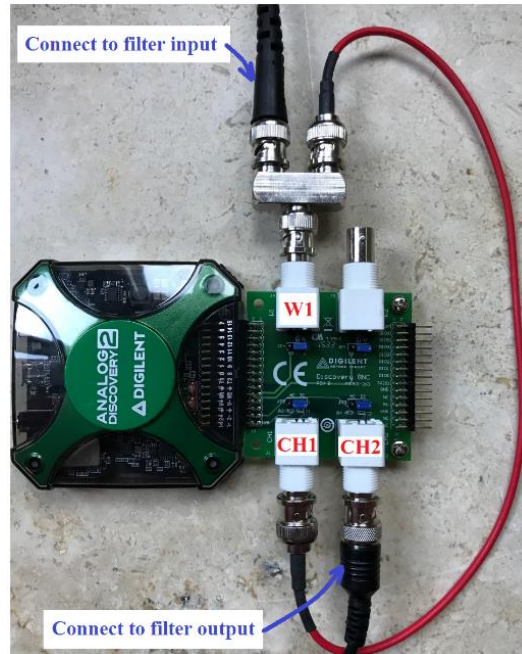


Note: The above formula for approximating the filter output does not take into consideration the filter’s phase shifting of the various harmonics. A more accurate approach is outlined in Step i.

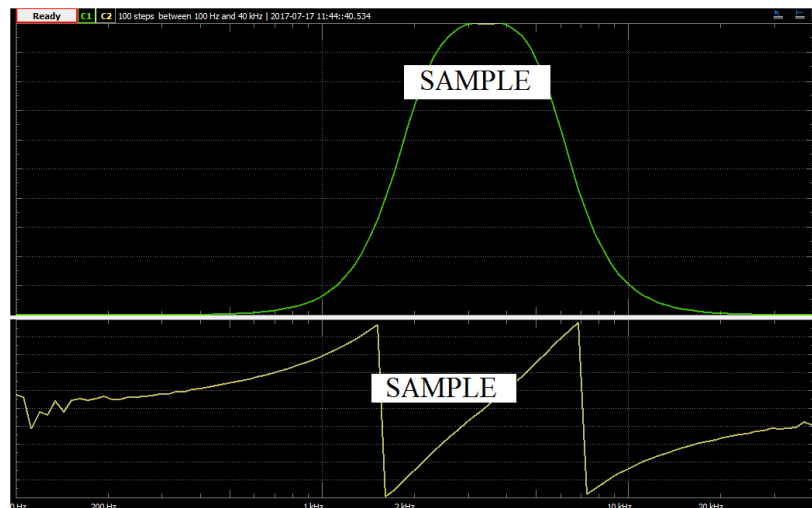
- h. Change the input signal from a square wave to a triangular wave (all other signal parameters and settings stay the same). Set CH2 to 200mV/div. Capture and save the image of the spectra of the input and the filtered signals from WaveForms. Also, save the RIGOL scope’s display image.

Exit WaveForms.

- i. In this final step you will employ the AD2/WaveForms setup to generate the magnitude and phase spectra (frequency response) of the band-pass filter used in Steps f and g. Disconnect the signals from the RIGOL waveform generator and the RIGOL scope. As you did in Experiment 6, set up the connections between the AD2 module and the band-pass filter (do not change the cutoff frequencies of the filter), as shown in the figure below.



Start WaveForms and select the “experiment7-filter” configuration file when prompted (you were asked to download this configuration file to the proper folder on your PC in the pre-lab activity). Click on “Single” run from the WaveForms screen. You should see a spectral plot that is similar to the one shown below. Capture the screen.



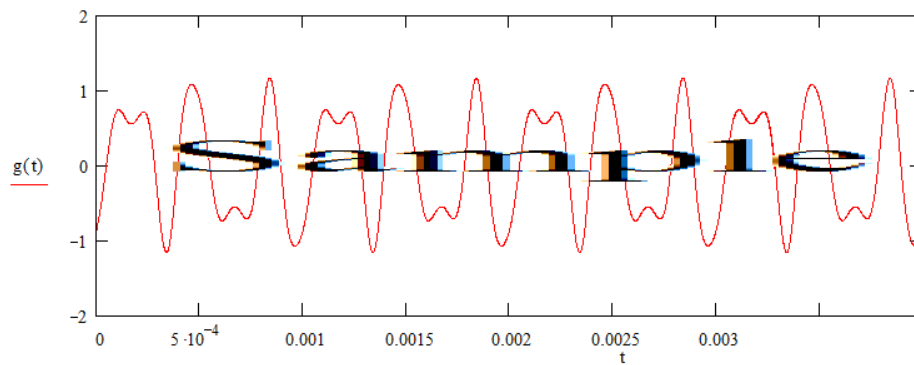
Measure the gain at 2kHz and 6kHz.

Determine (from the spectral plots) the frequency and phase at the peak of the magnitude response.

Measure the amplitude H_i and phase φ_i ($i = 1, 3, 5, 7, 9$) at the following frequencies: 1kHz, 3kHz, 5kHz, 7kHz and 9kHz. In your lab report, you are to plot the following expression for the filter output (and compare it to the those generated in Steps f and g),

$$g(t) \cong (H_1) \frac{8}{\pi} \cos((2\pi 1000)t + \varphi_1) + (H_3) \frac{8}{3\pi} \cos(3(2\pi 1000)t + \pi + \varphi_3) + \\ (H_5) \frac{8}{5\pi} \cos(5(2\pi 1000)t + \varphi_5) + (H_7) \frac{8}{7\pi} \cos(7(2\pi 1000)t + \pi + \varphi_7) \\ + (H_9) \frac{8}{9\pi} \cos(9(2\pi 1000)t + \varphi_9)$$

The phase φ_i must be in radian in the above expression. The following is a sample plot:



As part of the Post-lab activity, you are to compare the experimentally obtained filter's spectra to the one you compute (use Mathcad) by cascading (multiplying) the frequency response functions of a 4th order high-pass Butterworth filter with cutoff frequency of 2kHz and a 4th order low-pass Butterworth filter with cutoff frequency of 6kHz. Plot the magnitude and phase response (in degrees) for a logarithmic frequency axis ($100\text{Hz} \leq f \leq 10^5\text{Hz}$).

Hint: A 4th order high-pass Butterworth filter with cutoff frequency f_c has the frequency response function:

$$H_{HP}(f) = \frac{\left(\frac{f}{f_c}\right)^4}{\left(\frac{f}{f_c}\right)^4 - j2.61\left(\frac{f}{f_c}\right)^3 - 3.41\left(\frac{f}{f_c}\right)^2 + j2.61\left(\frac{f}{f_c}\right) + 1}$$

3. Post-lab

Write your lab report according to the required format. Analyze and discuss, in details, all results obtained from your pre-lab and your lab activity. Compare the theoretical and experimental performance of the filter. Report the absolute relative errors (in percent) for all experimental measurement whenever the true (ideal) value is known.